# Linear and Branching System Metrics

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***Abstract*— We extend the classical system relations of trace inclusion, trace equivalence, simulation, and bisimulation to a quantitative setting in which propositions are interpreted not as boolean values, but as elements of arbitrary metric spaces. Trace inclusion and equivalence give rise to asymmetrical and symmetrical *linear distances*, while simulation and bisimulation give rise to asymmetrical and symmetrical *branching distances*. We study the relationships among these distances, and we provide a full logical characterization of the distances in terms of quantitative versions of LTL and $\mu$-calculus. We show that, while trace inclusion (resp. equivalence) coincides with simulation (resp. bisimulation) for deterministic boolean transition systems, linear and branching distances do not coincide for deterministic metric transition systems. Finally, we provide algorithms for computing the distances over finite systems, together with a matching lower complexity bound.**

## I. Introduction

SOFTWARE verification tries to develop automatic tools for the analysis of correctness properties of software. Often, the aim is to check whether a piece of software, or an abstract model of it, conforms to a given specification. Classical techniques, such as model-checking, are only capable of yes-no replies: either the system meets its specification, or it does not. In contrast, in this paper we examine *quantitative* techniques for comparing a system with its specification. That is, we quantify to what extent a system meets its specification. To do so, we introduce and compare different ways to measure the distance between two systems. When two systems are at distance zero, they are indistinguishable w.r.t. some equivalence criterion (such as behavior step-wise simulation or behavior inclusion). While for safety-critical applications, any distance greater than zero signifies the presence of a catastrophic bug, in other cases small discrepancies may be tolerated, for instance to reduce the product costs. For example, consider an MP3 player. If the player is required to react within 1 second to user input, but does so within 1.05 seconds, this may in fact be a viable solution, even though the system does not meet its specification in the classical, boolean sense. In our setting, we would say that the distance from the player to its specification is 0.05 (on a scale where we consider deviations up to 1.0 seconds).

We conduct our analysis on a very general model, called *metric transition system*. A metric transition system is a transition system in which the propositions, at each state, are interpreted as elements of metric spaces. Many examples of metric transition systems have been studied in the literature. As the set $\mathbb{R}$ of real numbers is a metric space (when equipped, for instance, with the metric $d(x, y) = |x - y|$), hybrid systems (where clocks and hybrid variables are interpreted in $\mathbb{R}$) and priced automata (where a real-valued “price” is associated with each state) are all examples of metric transition systems. Kripke structures are also a special case of metric transition systems, as the set $\{\mathrm{T}, \mathrm{F}\}$ of boolean values can be associated with the metric $d(\mathrm{T}, \mathrm{T}) = d(\mathrm{F}, \mathrm{F}) = 0$, and $d(\mathrm{T}, \mathrm{F}) = d(\mathrm{F}, \mathrm{T}) = 1$. Indeed, almost all classes of transition systems that have been proposed in the literature constitute metric transition systems.

Trace inclusion, trace equivalence, simulation, and bisimulation are classical system relations which play a very important role in system specification and verification. These system relations are defined in terms of the *equality* of propositional valuations: for example, trace inclusion holds between two states $s$, $t$ if every trace from $s$ can be exactly matched, in terms of propositional valuations, by a trace from $t$. Once propositions are evaluated in metric spaces, the system relations themselves can be generalized to metrics. Thus, we propose to generalize trace inclusion to a *linear distance* that measures how closely a path from $s$ can be matched by a path from $t$, in terms of the *distance* between the corresponding propositional valuations. Following this idea, we extend the classical relations of trace inclusion, trace equivalence, simulation, and bisimulation to a metric setting, by defining linear and branching *distances*[1]. Considering distances, rather than relations, leads to a theory of system approximations [8], [17], [2]. In most engineering disciplines, specifications include information about the allowed *tolerance* (maximum deviation) in their implementation. The metrics proposed in this paper enable us to extend this approach to behavioral specifications, by capturing how closely the behavior of a concrete system implements a specification. Furthermore, for systems whose propositions are evaluated in dense metric spaces (such as $\mathbb{R}$), system metrics are often more meaningful than system relations, as they are robust with respect to perturbations in the propositional valuations. For instance, in system models whose parameters are determined via experimental observations subject to measurement errors, system metrics provide useful information about behavioral similarity, while system relations provide unnecessarily fine-grained, and ultimately meaningless, information.

We define two families of distances: *linear distances*, which generalize trace inclusion and equivalence, and *branching distances*, which generalize (bi)simulation. We relate these distances to the quantitative version of the two well-known specification languages LTL and $\mu$-calculus, showing that the distances measure to what extent the logic can tell one system from the other. The distance notions arising as generalizations of trace inclusion and simulation are asymmetrical, just like the relations they generalize: the “simulation distance” from $s$ to $t$ is in general different from the “simulation distance” from $t$ to $s$. We call these asymmetrical distances *directed metrics*,

[1] In this paper, we use the term “distance” in a generic way, applying it to various types of metrics.

preferring this term to the term *quasi-pseudometrics* used elsewhere in the literature [10]; symmetrical distances will be called *undirected metrics*. Thus, for the sake of generality, we develop our results in the general setting where propositions are evaluated in spaces endowed with directed metrics.

Our starting point for linear distances is the distance $\|\sigma - \rho\|_\infty$ between two traces $\sigma$ and $\rho$, which measures the supremum of the difference in propositional valuations at corresponding positions of $\sigma$ and $\rho$. To lift this trace distance to a distance over states, we define $ld^{\rm s}(s,t) = \sup_{\sigma \in Tr(s)} \inf_{\rho \in Tr(t)} \|\sigma - \rho\|_\infty$, where $Tr(s)$ and $Tr(t)$ are the set of traces from $s$ and $t$, respectively. The distance $ld^{\rm s}(s,t)$ is asymmetrical, and is a quantitative extension of trace containment: assuming that the system is finitely-branching, if $ld^{\rm s}(s,t) = b$, then for all traces $\sigma$ from $s$, there is a trace $\rho$ from $t$ such that $\|\sigma - \rho\|_\infty \leq b$. In particular, if the metric spaces where the propositions are evaluated assign distance 0 only to identical elements, then $Tr(s) \subseteq Tr(t)$ iff $ld^{\rm s}(s,t) = 0$. We define a symmetrical version of this distance by $\overline{ld}^{\rm s}(s,t) = \max\{ld^{\rm s}(s,t), ld^{\rm s}(t,s)\}$, yielding a distance that generalizes trace equivalence; thus, $\overline{ld}^{\rm s}(s,t)$ is the Hausdorff distance between $Tr(s)$ and $Tr(t)$.

We relate the linear distances to the logic QLTL, a quantitative version of LTL [13]. When interpreted on a metric transition system, QLTL formulas yield a value in the positive reals. The propositional formulas of QLTL are of the form $D(r,c)$ and $D(c,r)$, where $r$ is a proposition, and $c$ is a constant denoting an element of the same metric space where $r$ is evaluated. The formula $D(r,c)$, at a state, yields the distance of the valuation of $r$ at the state from the constant $c$. Both $D(r,c)$ and $D(c,r)$ are present as basic formulas: in our setting based on directed metrics, the distance from the valuation of $r$ to $c$, and the distance from $c$ to the valuation of $r$, need not be the same. The formula "next $p$" returns the (quantitative) value of the subformula $p$ in the next step of a trace, while "eventually $p$" seeks the maximum value attained by $p$ throughout the trace. The logical connectives "and" and "or" are interpreted as "min" and "max."

In the standard relational setting, for a relation to characterize a logic, two states must be related if and only if all formulas from the logic have the same truth value on them. In our metric framework, we can achieve a finer characterization: in addition to relating those states that formulas cannot distinguish, we can also *measure* to what extent the logic can tell one state from the other. We give two kinds of characterizations. We show that for arbitrary metric transition systems, the distances provide a bound for the difference in value of QLTL formulas: precisely, for all states $s,t$ and QLTL formulas $\varphi$ we have $|\varphi(t) - \varphi(s)| \leq \overline{ld}^{\rm s}(s,t)$ and $\varphi(t) - \varphi(s) \leq ld^{\rm s}(s,t)$. Moreover, we show that for finitely branching metric transition systems, such characterizations are tight: for all states $s,t$ we have $\overline{ld}^{\rm s}(s,t) = \sup_{\varphi \in {\rm QLTL}} |\varphi(t) - \varphi(s)|$ and $ld^{\rm s}(s,t) = \sup_{\varphi \in {\rm QLTL}} (\varphi(t) - \varphi(s))$. This tightness result does not hold in general for non-finitely-branching metric transition systems.

We then study the branching distances that are the analogue of simulation and bisimulation on quantitative systems. Recall that a state $s$ simulates a state $t$ via a relation $R$ if the propositional valuations at $s$ and $t$ coincide, and if every successor of $s$ is related via $R$ to some successor of $t$. We generalize simulation to a distance $bd^{\rm As}$ over states. If $bd^{\rm As}(s,t) = b$, then the valuations of corresponding propositions at $s$ and $t$ differ by at most $b$, and every successor of $s$ can be matched by a successor of $t$ within $bd^{\rm As}$-distance $b$. In a similar fashion, we can define a distance $bd^{\rm Ss}$ that is a quantitative analogue of bisimulation; such a distance has been studied in [8], [17]. We relate these distances to QMU, a quantitative fixpoint calculus that closely resembles the $\mu$-calculus of [4], and is related to the calculi of [12], [5] (see also [11], [14]). Similarly to QLTL, the basic formulas of QMU are of the form $D(r,c)$ and $D(c,r)$, for a proposition $r$ and a valuation $c$. The modal formulas $\forall$ $p$, $\exists$ $p$ compute respectively the least and greatest value of a subformula $p$ at all successor states; the logical connectives "and" and "or" are interpreted as "min" and "max", and the fixpoints are given a quantitative interpretation.

Again, we provide a twofold logical characterization of the branching distances in terms of QMU. We show that for arbitrary metric transition systems, we have $|\varphi(t) - \varphi(s)| \leq bd^{\rm Ss}(s,t)$ and $\psi(t) - \psi(s) \leq bd^{\rm As}(s,t)$, where $\varphi$ is any QMU-formula, and $\psi$ is any "universal" QMU-formula, i.e., any formula of QMU that does not contain $\exists$ . Moreover, if the metric transition system is finitely branching, then we have the stronger result $bd^{\rm Ss}(s,t) = \sup_{\varphi \in {\rm QMU}} |\varphi(t) - \varphi(s)|$ and $bd^{\rm As}(s,t) = \sup_{\psi \in \exists{\rm QMU}} (\psi(t) - \psi(s))$, where $\exists$QMU is the fragment of QMU in which $\exists$ does not occur; these results do not hold in general for non-finitely-branching metric transition systems.

We relate linear and branching distances, showing that just as simulation implies trace containment, so the branching distances are greater than or equal to the corresponding linear distances. However, we show that determinism plays a lesser role in the quantitative setting than in the standard boolean setting: while trace inclusion (resp. equivalence) coincides with simulation (resp. bisimulation) for deterministic boolean transition systems, we show that linear and branching distances do not coincide for deterministic metric transition systems. Finally, we present algorithms for computing linear and branching distances over metric transition systems. We show that the problem of computing the linear distances is PSPACE-complete, and it remains PSPACE-complete even over deterministic systems, showing once more that determinism plays a lesser role in the quantitative setting. The branching distances can be computed in polynomial time using standard fixpoint algorithms, similarly to [4].

We extend all our results to a *discounted* context, in which distances occurring after $i$ steps in the future are multiplied by $\alpha^i$, where $\alpha$ is a discount factor in $[0,1]$. This discounted setting is common in the theory of games (see e.g. [9]) and optimal control (see e.g. [7]), and it leads to robust theories of quantitative systems [4]. In the discounted setting, behavioral differences arising far into the future are given less relative weight than behavioral differences affecting the present or the near future. Hence, the discounted setting leads to notions of "local similarity" that enjoy many pleasant mathematical properties.

## II. PRELIMINARIES

We denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{R}_+$ the set of non-negative reals. For two numbers $x, y \in \mathbb{R}$, we write $x \sqcup y = \max(x, y)$ and $x \sqcap y = \min(x, y)$. We lift the operators $\sqcup$ and $\sqcap$, and the relations $<, \leq$ to functions via their pointwise extensions. Precisely, for $n$-argument functions $f_1, f_2 : A_1 \times \cdots \times A_n \to \mathbb{R}$, we write $f_1 \sqcup f_2$ for the function $g : A_1 \times \cdots \times A_n \to \mathbb{R}$ defined by $g(x_1, \ldots, x_n) = f_1(x_1, \ldots, x_n) \sqcup f_2(x_1, \ldots, x_n)$, and similarly for $\sqcap$; we write $f_1 \leq f_2$ if $f_1(x_1, \ldots, x_n) \leq f_2(x_1, \ldots, x_n)$ for all $x_1 \in A_1, \ldots, x_n \in A_n$, and we write $f_1 < f_2$ if $f_1 \leq f_2$ and if there are some $x_1 \in A_1, \ldots, x_n \in A_n$ for which $f_1(x_1, \ldots, x_n) < f_2(x_1, \ldots, x_n)$. Given a function $d : X^2 \to \mathbb{R}$, we denote by $\mathrm{Zero}(d) = \{(x, y) \in X^2 \mid d(x, y) = 0\}$ its zero set. Given a sequence $\{x_i\}_{i \in \mathbb{N}}$, we commonly write $\lim_i x_i$ for $\lim_{i \to \infty} x_i$. The following lemma summarizes some simple facts about sequences of real numbers that will be needed in subsequent proofs.

*Lemma 1:* *Let $\mathcal{I}$ be a set and $\{x_i\}_{i \in \mathcal{I}}$, $\{y_i\}_{i \in \mathcal{I}}$ be two families of numbers in $\mathbb{R}$. The following assertions hold.*

1) *If $x_i - y_i \leq c$ for all $i \in \mathcal{I}$, then $\sup_i x_i - \sup_i y_i \leq c$ and $\inf_i x_i - \inf_i y_i \leq c$.*
2) *Let $X, Y$ be sets and $f : X \times Y \to \mathbb{R}$ be a function. Then*

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) \leq \inf_{y \in Y} \sup_{x \in X} f(x, y).$$

■

### A. Metrics and Metric Spaces

We define *directed and undirected metrics,* where undirected metrics are required to be symmetrical and directed metrics are not. For example, the travel distance between two points in a city with one-way streets is a directed metric. Our directed and undirected metrics generalize the usual metrics, in that elements that have metric 0 are not required to be identical. The definitions are as follows.

*Definition 1:* **(metrics)** We introduce the following terminology.

1) A *directed metric* on a set $X$ is a function $d : X \times X \to \mathbb{R}$ that satisfies
   - $d(x, x) = 0$ for all $x \in X$;
   - $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (triangle inequality).

   A directed metric $d$ is *proper* if $d(x, y) = 0$ implies $x = y$ (identity of indiscernibles).
2) An *undirected metric* is a directed metric $d : X \times X \to \mathbb{R}$ that is symmetrical, that is, such that $d(x, y) = d(y, x)$ for all $x, y \in X$. Undirected metrics are also called simply *metrics.* ■

We will often define a directed metric, and obtain the corresponding undirected metric by *symmetrization.*

*Definition 2:* **(symmetrization)** Given a directed metric $d$ on a set $X$, we denote by $\bar{d}$ its *symmetrization,* defined by $\bar{d}(x, y) = d(x, y) \sqcup d(y, x)$ for all $x, y \in X$. Obviously, for all $x, y \in X$, we have $d(x, y) \leq \bar{d}(x, y)$. ■

In a Kripke structure, the value of a proposition at each state is a member of the truth-value set $\{\mathrm{T}, \mathrm{F}\}$. We extend this setting by evaluating propositions, at each state, to elements of *metric spaces.* A metric space is a set with a metric defined on it; for the sake of generality, we assume only that the metric is a directed metric.

*Definition 3:* **(directed metric space)** A *directed metric space*, or shortly a metric space, is a pair $(X, d)$, where $d$ is a directed metric on $X$. ■

We say that a metric space $(X, d)$ is *bounded* if the maximum distance between any two elements of $X$ is finite.

*Example 1:* An example of metric space is the space of RGB-represented colors, where the distance between colors $c_1$ and $c_2$ represents the difference in brightness between $c_1$ and $c_2$. The space is then $X = [0, 1]^3$, and for $\vec{x} = \langle x_1, x_2, x_3 \rangle$ and $\vec{y} = \langle y_1, y_2, y_3 \rangle$ we define $d(\vec{x}, \vec{y}) = |\vec{x} \cdot \vec{b} - \vec{y} \cdot \vec{b}|$, where $\vec{b}$ is a vector giving the brightness of each basic color, and $\cdot$ is the internal product. It is easy to see that $(X, d)$ is a bounded directed metric space. In particular, $d$ is undirected and not proper, as different colors may have the same brightness. ■

*Example 2:* Another example of a metric space is $\mathbf{X}_{\mathbb{R}} = (\mathbb{R}, d_{\mathbb{R}})$, with $d_{\mathbb{R}}(x, y) = \max\{x - y, 0\}$ for $x, y \in \mathbb{R}$. It is immediate that $d_{\mathbb{R}}$ is a directed metric and that $\mathbf{X}_{\mathbb{R}}$ is not bounded. On the other hand, the metric space $\mathbf{X}_{[0,1]} = ([0, 1], d_{\mathbb{R}})$ is bounded. ■

*Example 3:* A particularly simple example of bounded metric space is $\mathbf{X}_{\mathbb{B}} = (X, d_{\mathbb{B}})$, where $X = \{0, 1\}$ and $d(x, y) = |x - y|$ for $x, y \in \{0, 1\}$. This is the usual space of "boolean" valuations; it is immediate that $d$ is an undirected metric. ■

When providing logical characterizations for the distances, we will first consider logics in which any element of the metric space can be used as a constant. If the metric space is uncountable, however, this leads to the consideration of logics with uncountably many symbols. If a metric space is *separable*, however, each element can be approximated by arbitrarily close elements of a *countable basis.* In this case, we will see that logics with countably many symbols (corresponding to the elements of the basis) will suffice.

*Definition 4:* **(separable directed metric space)** A directed metric space $(X, d)$ is *separable* if there is a countable *basis* $\mathcal{B} \subseteq X$ such that, for all $x \in X$ and all $\varepsilon > 0$, there is $y \in \mathcal{B}$ with $d(x, y) < \varepsilon$ and $d(y, x) < \varepsilon$. ■

### B. Metric Transition Systems

A *metric transition system* is a transition system where the value of a proposition, at each state, is an element of a bounded directed metric space. To simplify the notation, we assume throughout the paper an underlying set *AP* of propositions, where each proposition $r \in AP$ takes values in a bounded metric space $(X_r, d_r)$.

*Definition 5:* **(valuations)** A *valuation* $u$ of a set $\Sigma \subseteq AP$ of propositions is a function with domain $\Sigma$ that assigns to each $r \in \Sigma$ an element $x \in X_r$ of the metric space $(X_r, d_r)$

corresponding to $r$. We denote by $\mathcal{U}[\Sigma]$ the set of all valuations of $\Sigma$. ■

*Definition 6:* **(metric transition system)** A *metric transition system* (MTS) is a tuple $M = (S, \tau, \Sigma, [\cdot])$ consisting of the following components:

- a set $S$ of states;
- a transition relation $\tau \subseteq S \times S$;
- a finite set $\Sigma \subseteq AP$ of propositions;
- a function $[\cdot]$: $S \to \mathcal{U}[\Sigma]$ that assigns to each state $s \in S$ a valuation $[s]$.

For a state $s \in S$, we write $\tau(s)$ for $\{t \in S \mid (s,t) \in \tau\}$. We require that $M$ is non-blocking: for all $s \in S$, the set $\tau(s)$ is non-empty. ■

We distinguish the following special classes of MTSs.

*Definition 7:* **(special types of MTSs)** Let $M = (S, \tau, \Sigma, [\cdot])$ be an MTS.

- We say that $M$ is *finite* if $S$ is finite.
- We say that $M$ is *deterministic* if for all states $s \in S$ and $t, t' \in \tau(s)$ with $t \neq t'$, there is $r \in \Sigma$ such that $[t](r) \neq [t'](r)$.
- We say that $M$ is *finitely branching* if $\tau(s)$ is finite for all $s \in S$.
- We say that $M$ is *separable* if, for all $r \in \Sigma$, the metric space $(X_r, d_r)$ is separable. In this case, we denote by $\mathcal{B}_r$ a countable basis for $(X_r, d_r)$. ■

### C. Paths and Traces

Given a set $A$ and a sequence $\pi = a_0 a_1 a_2 \cdots \in A^\omega$, we write $\pi_i$ for the $i$-th element $a_i$ of $\pi$, and we write $\pi^i = a_i a_{i+1} a_{i+2} \cdots$ for the (infinite) suffix of $\pi$ starting from $\pi_i$.

*Definition 8:* **(paths and traces)** Consider an MTS $M = (S, \tau, \Sigma, [\cdot])$. A *path* of $M$ is an infinite sequence of states $\pi \in S^\omega$ such that $(\pi_i, \pi_{i+1}) \in \tau$ for all $i \in \mathbb{N}$. Given a state $s \in S$, we write $\mathit{Paths}_M(s)$ for the set of all paths of $M$ starting from $s$; we omit the subscript $M$ when clear from the context.

A *trace* is an infinite sequence $\sigma \in \mathcal{U}[\Sigma]^\omega$. Every path $\pi$ of $M$ induces a trace $[\pi] = [\pi_0][\pi_1][\pi_2] \cdots$. We write $\mathit{Tr}_M(s) = \{[\pi] \mid \pi \in \mathit{Paths}_M(s)\}$ for the set of traces of $M$ starting from the state $s \in S$, and we omit the subscript $M$ when clear from the context. ■

### D. Branching and Trace Relations

We define simulation, bisimulation, trace containment, and trace equivalence for MTSs as usual.

*Definition 9:* **((bi)simulation, trace containment and trace equivalence)** For an MTS $M = (S, \tau, \Sigma, [\cdot])$, the simulation relation $\preceq_{\mathit{sim}}$ (resp. the bisimulation relation $\approx_{\mathit{bis}}$) is the largest relation $R \subseteq S \times S$ such that, for all $s \, R \, t$, the following Conditions 1 and 2 (resp. 1, 2, and 3) hold:

1) $[s] = [t]$;
2) for all $s' \in \tau(s)$, there is $t' \in \tau(t)$ with $s' \, R \, t'$;
3) for all $t' \in \tau(t)$, there is $s' \in \tau(s)$ with $s' \, R \, t'$.

For $s, t \in S$, we write $s \sqsubseteq_{\mathit{tr}} t$ if $\mathit{Tr}(s) \subseteq \mathit{Tr}(t)$, and $s \equiv_{\mathit{tr}} t$ if $\mathit{Tr}(s) = \mathit{Tr}(t)$. ■

### E. Discussion

We note that, for some of the results on system metrics, it would have been sufficient to define a metric transition system as a system that maps each state into an element of a metric space, bypassing thus the introduction of a set of propositions, and the related machinery. Such a definition, of course, is a special case of the one we adopt, and corresponds to considering metric transition systems with only one proposition. The main function of propositions is to enable us to develop the connection between system metrics and logics, since the logics refer to quantities via the propositions.

In an MTS $(S, \tau, \Sigma, [\cdot])$, we call each $r \in \Sigma$ a "proposition", rather than "variable", in spite of the fact that $r$ takes values in a generic metric space $(X_r, d_r)$, rather than in the set of truth-values. Our choice of terminology is motivated by the fact that in the system logics we consider, the symbol $r$ plays a (syntactic) role that is analogous to that of ordinary propositions. We reserve instead the term "variable" for the variables used to construct fixpoint expressions in $\mu$-calculus.

## III. Linear Distances and Logics

### A. Linear Distances

Throughout the paper, unless specifically noted, we consider a fixed MTS $M = (S, \tau, \Sigma, [\cdot])$. We proceed by defining the linear distances between valuations, then between traces and finally between states. The propositional distance between two valuations is the maximum difference in their proposition evaluations, where differences in the assignments of proposition $r$ are measured by the metric $d_r$.

*Definition 10:* **(propositional distance)** We define the *propositional distance* $pd : \mathcal{U}[\Sigma]^2 \to \mathbb{R}$, for all valuations $u, v \in \mathcal{U}[\Sigma]$, as $pd(u,v) = \max_{r \in \Sigma} d_r(u(r), v(r))$. ■

For ease of notation, we write $pd(s,t)$ for $pd([s],[t])$. If all $\Sigma$-metrics are proper, then given $u, v \in \mathcal{U}[\Sigma]$ we have $(u,v) \in \mathrm{Zero}(pd)$ iff $u = v$.

*Example 4:* Consider states $s_4$ and $t_4$ in Figure 1, where proposition $r$ is evaluated in the metric space $\mathbf{X}_{[0,1]}$. Then $pd(s_4, t_4) = 0$, $pd(t_4, s_4) = 0.3$, and $\overline{pd}(s_4, t_4) = 0.3$. ■

The trace distance is the pointwise extension of the propositional distance to infinite sequences of valuations.

*Definition 11:* **(trace distance)** We define the *trace distance* $td : \mathcal{U}[\Sigma]^\omega \times \mathcal{U}[\Sigma]^\omega \to \mathbb{R}$ by letting, for $\sigma, \rho \in \mathcal{U}[\Sigma]^\omega$, $td(\sigma, \rho) = \sup_{i \in \mathbb{N}} pd(\sigma_i, \rho_i)$. ■

*Example 5:* Consider the states $s_0$ and $t_0$ in Figure 1. Both contain two traces: let $\sigma_0 = s_0 s_1 s_3^\omega$ and $\sigma_1 = s_0 s_1 s_4^\omega$ denote respectively the leftmost and rightmost trace from $s_0$; let $\rho_0 = t_0 t_1 t_3^\omega$ and $\rho_1 = t_0 t_2 t_4^\omega$ denote the leftmost and rightmost trace from $t_0$. Then

$$
\begin{aligned}
td(\sigma_0, \rho_0) &= 0 & \overline{td}(\sigma_0, \rho_0) &= 0.1 \\
td(\sigma_0, \rho_1) &= 0 & \overline{td}(\sigma_0, \rho_1) &= 0.6 \\
td(\sigma_1, \rho_0) &= 0.2 & \overline{td}(\sigma_1, \rho_0) &= 0.2 \\
td(\sigma_1, \rho_1) &= 0 & \overline{td}(\sigma_1, \rho_1) &= 0.3.
\end{aligned}
$$

■

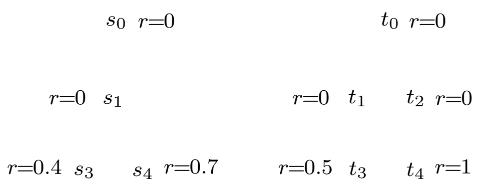



Fig. 1. MTS illustrating the linear distances. Proposition $r$ is evaluated in the metric space $\mathbf{X}_{[0,1]}$.

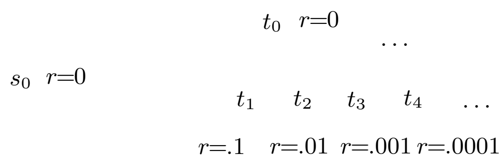



Fig. 2. An infinitely branching MTS showing the difference between $\mathrm{Zero}(ld^{\mathrm{s}})$ and $\sqsubseteq_{tr}$. Proposition $r$ is evaluated in the metric space $\mathbf{X}_{[0,1]}$.

It is easy to show that $td$ is a directed metric. The following result states that if we base the notion of trace distance on $\overline{pd}$ instead of on $pd$ (i.e. if we replace $pd$ by $\overline{pd}$ in the definition above), we obtain the symmetrization $\overline{td}$ of $td$. Moreover, the kernel of this symmetrization is trace equality.

*Lemma 2: For all sequences $\sigma$, $\rho \in \mathcal{U}[\Sigma]^\omega$, we have $\overline{td}(\sigma,\rho) = \sup_{i\in\mathbb{N}} \overline{pd}(\sigma_i,\rho_i)$. Moreover, if $d_r$ is a proper metric for all $r \in \Sigma$, then $(\sigma,\rho) \in \mathrm{Zero}(\overline{td})$ if and only if $\sigma = \rho$.*

The linear distances between two states are obtained by lifting the trace distances to the sets of traces emerging from those states, as in the definition of the Hausdorff distance between sets.

The intuition is as follows. To establish trace inclusion between states $s$ and $t$, we check if, for a trace from $s$, the same trace exists from $t$. If there is a trace from $s$ that cannot be matched from $t$, there is no trace inclusion.

For the linear distance, we match each trace $\sigma$ from $s$ with the trace $\rho$ from $t$ with the smallest trace distance to $\sigma$ (or the infimum of these $\rho$'s if the minimum is not attained). This yields distance $\inf_{\rho\in Tr(t)} \overline{td}(\sigma,\rho)$ for $\sigma$. Then, we consider the trace from $s$ that is the hardest to match, yielding distance $\sup_{\sigma\in Tr(s)} \inf_{\rho\in Tr(t)} \overline{td}(\sigma,\rho)$.

*Definition 12:* **(linear distance)** We define the two *linear distances* $ld^{\mathrm{a}}$ and $ld^{\mathrm{s}}$ over $S$ by letting, for all $s,t \in S$

$$ld^{\mathrm{a}}(s,t) = \sup_{\sigma\in Tr(s)} \inf_{\rho\in Tr(t)} td(\sigma,\rho)$$
$$ld^{\mathrm{s}}(s,t) = \sup_{\sigma\in Tr(s)} \inf_{\rho\in Tr(t)} \overline{td}(\sigma,\rho). \quad \square$$

One can easily check that the functions $ld^{\mathrm{a}}$ and $ld^{\mathrm{s}}$ are directed metrics, while $\overline{ld}^{\mathrm{a}}$ and $\overline{ld}^{\mathrm{s}}$ are undirected ones. Intuitively, the distance $ld^{\mathrm{s}}$ is a quantitative extension of trace containment: for $s,t \in S$, the distance $ld^{\mathrm{s}}(s,t)$ measures how closely (in a quantitative sense) a trace from $s$ can be simulated by a trace from $t$. The symmetrization of $ld^{\mathrm{s}}$ is $\overline{ld}^{\mathrm{s}}$, which is related to trace equivalence. Indeed, we will see in the next section that it is possible to define a quantitative logic QLTL such that the valuation of QLTL formulas at $s$ and $t$ can differ by at most $\overline{ld}^{\mathrm{s}}(s,t)$, and similarly, the valuation of any QLTL formula at $t$ is at most $ld^{\mathrm{s}}(s,t)$ below the valuation at $s$.

*Example 6:* We write $ld^{\mathrm{a}}(\sigma,t)$ for $\inf_{\rho\in Tr(t)} td(\sigma,\rho)$ and similarly for $ld^{\mathrm{s}}(\sigma,t)$. Using the trace distances computed in Example 5, we obtain for the MTS in Figure 1

$$ld^{\mathrm{a}}(\sigma_0,t_0) = td(\sigma_0,\rho_0) \sqcap td(\sigma_0,\rho_1) = 0 \sqcap 0 = 0$$
$$ld^{\mathrm{a}}(\sigma_1,t_0) = td(\sigma_1,\rho_0) \sqcap td(\sigma_1,\rho_1) = 0.2 \sqcap 0 = 0.$$

We obtain that $ld^{\mathrm{a}}(s_0,t_0) = ld^{\mathrm{a}}(\sigma_0,t_0) \sqcup ld^{\mathrm{a}}(\sigma_1,t_0) = 0$. Similarly,

$$ld^{\mathrm{s}}(\sigma_0,t_0) = \overline{td}(\sigma_0,\rho_0) \sqcap \overline{td}(\sigma_0,\rho_1) = 0.1 \sqcap 0.6 = 0.1$$
$$ld^{\mathrm{s}}(\sigma_1,t_0) = \overline{td}(\sigma_1,\rho_0) \sqcap \overline{td}(\sigma_1,\rho_1) = 0.2 \sqcap 0.3 = 0.2,$$

so that $ld^{\mathrm{s}}(s_0,t_0) = ld^{\mathrm{s}}(\sigma_0,t_0) \sqcup ld^{\mathrm{s}}(\sigma_1,t_0) = 0.2$. ■

*Example 7:* Consider the case where $(X_r,d_r) = \mathbf{X}_{[0,1]}$ for all $r \in \Sigma$, that is, all propositions are interpreted as real numbers in the interval $[0,1]$, and $d_r(a,b)$ is a measure of how much greater is $a$ than $b$. In this setting, the distances $ld^{\mathrm{a}}$ and $\overline{ld}^{\mathrm{a}}$ have the following intuitive characterization. For $x,y \in [0,1]$, let $x \dot{-} y = \max\{x-y,0\}$. For a trace $\sigma \in \mathcal{U}[\Sigma]^\omega$ and $c \in \mathbb{R}$, denote by $\sigma \dot{-} c$ the trace defined by $(\sigma \dot{-} c)_k(r) = \sigma_k(r) \dot{-} c$ for all $k \in \mathbb{N}$ and $r \in \Sigma$: in other words, $\sigma \dot{-} c$ is obtained from $\sigma$ by decreasing all propositional valuations by $c$. Assuming that the system is finitely branching, for all $s,t \in S$, if $ld^{\mathrm{a}}(s,t) = c$ then for every trace $\sigma$ from $s$ there is a trace $\rho$ from $t$ such that $\rho \geq \sigma \dot{-} c$. This means that $ld^{\mathrm{a}}(s,t)$ is a "positive" version of trace containment: for each trace $\sigma$ of $s$, the goal of a trace $\rho$ from $t$ is not that of being close to $\sigma$, but rather, that of not being below $\sigma \dot{-} c$. Such an interpretation is important in a setting where values denote costs; thus, a system implementation whose costs are lower than specified lays at distance 0 from its specification. ■

*Theorem 1: For all finitely branching MTSs $(S,\tau,\Sigma,[\cdot])$, such that $d_r$ is a proper metric for all $r \in \Sigma$, we have $\sqsubseteq_{tr} = \mathrm{Zero}(ld^{\mathrm{s}})$ and $\equiv_{tr} = \mathrm{Zero}(\overline{ld}^{\mathrm{s}})$.*

*Proof:* Let $(S,\tau,\Sigma,[\cdot])$ be an MTS with $s,t \in S$. It is easy to see that $s \sqsubseteq_{tr} t$ implies $ld^{\mathrm{s}}(s,t) = 0$. To prove the converse, assume that $ld^{\mathrm{s}}(s,t) = 0$ and let $\sigma \in Tr(s)$. Then, there are traces $\rho_0,\rho_1,\rho_2 \ldots \in Tr(t)$ such that $\overline{td}(\sigma,\rho_i) < \frac{1}{2^i}$ for all $i$. Due to the finitely branching property, there exists a trace $\rho^*$ such that $\overline{td}(\sigma,\rho^*) < \frac{1}{2^i}$ for all $i$. This means that $\overline{td}(\sigma,\rho^*) = 0$, which, by Lemma 2, is the same as $\sigma = \rho^*$. Now, the result for $\equiv_{tr}$ and $\overline{ld}^{\mathrm{s}}$ easily follows. ■

To show that the result above does not hold for infinitely branching systems, consider the MTS in Figure 2, where the proposition $r$ is again evaluated in the metric space $\mathbf{X}_{[0,1]}$. This MTS has infinitely many states $s_0, t_0, t_1, t_2, \ldots$ and transitions $(s_0,s_0)$, $(t_0,t_i)$ and $(t_i,t_i)$ for each $i \in \mathbb{N}$. Moreover, we put $[s_0](r) = [t_0](r) = 0$ and $[t_i](r) = 10^{-i}$ for $i > 0$. Then, we have that $(s_0,t_0) \in \mathrm{Zero}(ld^{\mathrm{s}})$, but $s_0 \not\sqsubseteq_{tr} t_0$. To obtain an MTS with $\overline{ld}^{\mathrm{s}}(t_0,u_0) = 0$, but $t_0 \not\equiv_{tr} u_0$, we let $u_0$ be a state that is the exactly same as $t_0$ (i.e. same valuation and same successor states), except that it has a self-loop (i.e. a transition $(u_0,u_0) \in \tau$).

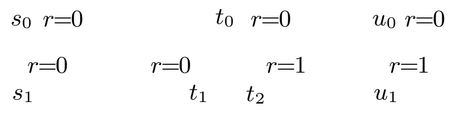



Fig. 3. An MTS showing the difference between $ld^{\mathrm{a}}$, $ld^{\mathrm{s}}$, $\overline{ld}^{\mathrm{a}}$, and $\overline{ld}^{\mathrm{s}}$. Proposition $r$ is evaluated in the metric space $\mathbf{X}_{[0,1]}$.

The relations among linear distances are stated by the following theorem, and summarized in Figure 6(a).

*Theorem 2: The following assertions hold.*

1) *For all MTSs, we have* $ld^{\mathrm{a}} \le \overline{ld}^{\mathrm{a}}$, $ld^{\mathrm{a}} \le ld^{\mathrm{s}}$, $ld^{\mathrm{s}} \le \overline{ld}^{\mathrm{s}}$, *and* $\overline{ld}^{\mathrm{a}} \le \overline{ld}^{\mathrm{s}}$. *Moreover, the inequalities cannot be replaced by equalities.*
2) *The distances* $ld^{\mathrm{s}}$ *and* $\overline{ld}^{\mathrm{a}}$ *are incomparable: there is an MTS with states* $s, t, z \in S$ *such that* $ld^{\mathrm{s}}(s,t) < \overline{ld}^{\mathrm{a}}(s,t)$ *and* $ld^{\mathrm{s}}(t,z) > \overline{ld}^{\mathrm{a}}(t,z)$.

*Proof:* The first and third inequalities of statement (1) are trivial, while the second and fourth follow immediately from the fact that, for all traces $\sigma$ and $\rho$, $td(\sigma,\rho) \le \overline{td}(\sigma,\rho)$. For the MTS in Figure 3, we have

$$
\begin{array}{lll}
ld^{\mathrm{a}}(s_0,t_0)=0 & ld^{\mathrm{a}}(t_0,u_0)=0 & ld^{\mathrm{a}}(u_0,t_0)=0\\
ld^{\mathrm{s}}(s_0,t_0)=0 & ld^{\mathrm{s}}(t_0,u_0)=1 & ld^{\mathrm{s}}(u_0,t_0)=0\\
\overline{ld}^{\mathrm{a}}(s_0,t_0)=1 & \overline{ld}^{\mathrm{a}}(t_0,u_0)=0 & \overline{ld}^{\mathrm{a}}(u_0,t_0)=0\\
\overline{ld}^{\mathrm{s}}(s_0,t_0)=1 & \overline{ld}^{\mathrm{s}}(t_0,u_0)=1 & \overline{ld}^{\mathrm{s}}(u_0,t_0)=1.
\end{array}
$$

Thus, we have an example where $ld^{\mathrm{a}} \ne ld^{\mathrm{s}}$, $ld^{\mathrm{a}} \ne \overline{ld}^{\mathrm{a}}$, $ld^{\mathrm{s}} \ne \overline{ld}^{\mathrm{s}}$, $\overline{ld}^{\mathrm{a}} \ne \overline{ld}^{\mathrm{s}}$, and neither $ld^{\mathrm{s}} \le \overline{ld}^{\mathrm{a}}$ nor $ld^{\mathrm{s}} \ge \overline{ld}^{\mathrm{a}}$. ■

Next, we show that the linear distances are robust with respect to perturbations in the state valuations: small changes in the propositional valuations causes small changes in the distances. Given two state valuations $[\cdot]_1, [\cdot]_2 : S \to \mathcal{U}[\Sigma]$, we define their distance by:

$$d([\cdot]_1,[\cdot]_2) = \sup_{s\in S}\max_{r\in\Sigma} d_r([s]_1(r),[s]_2(r)).$$

Moreover, for a state valuation $f : S \to \mathcal{U}[\Sigma]$, we write $ld^{\mathrm{a}}_f$, $ld^{\mathrm{s}}_f$ for the distances defined as in Definition 12, using $f$ as the state valuation.

*Theorem 3:* **(linear distance robustness)** *For all propositional valuations* $[\cdot]_1, [\cdot]_2$, *and all* $s,t \in S$, *we have*

$$
\begin{array}{l}
ld^{\mathrm{a}}_{[\cdot]_1}(s,t) - ld^{\mathrm{a}}_{[\cdot]_2}(s,t) \le d([\cdot]_1,[\cdot]_2) + d([\cdot]_2,[\cdot]_1)\\
ld^{\mathrm{s}}_{[\cdot]_1}(s,t) - ld^{\mathrm{s}}_{[\cdot]_2}(s,t) \le d([\cdot]_1,[\cdot]_2) + d([\cdot]_2,[\cdot]_1).
\end{array}
$$

*Proof:* The result follows by showing that the trace distance between two traces $\rho$ and $\sigma$, measured under $[\cdot]_1$ and $[\cdot]_2$, differs by at most $d([\cdot]_1,[\cdot]_2) + d([\cdot]_2,[\cdot]_1)$. The key step consists in noting that, for any $r \in \Sigma$, from the triangular inequality

$$
\begin{aligned}
d_r([s]_1(r),[t]_1(r)) \le\; & d_r([s]_1(r),[s]_2(r))\\
&+ d_r([s]_2(r),[t]_2(r))\\
&+ d_r([t]_2(r),[t]_1(r))
\end{aligned}
$$

follows

$$
\begin{aligned}
& d_r([s]_1(r),[t]_1(r)) - d_r([s]_2(r),[t]_2(r))\\
& \le d_r([s]_1(r),[s]_2(r)) + d_r([t]_2(r),[t]_1(r))\\
& \le d([\cdot]_1,[\cdot]_2) + d([\cdot]_2,[\cdot]_1).
\end{aligned}
$$

Now the result follows by repetitive application of Lemma 1(1). ■

## B. Quantitative Linear-Time Temporal Logic

The linear distances introduced above can be characterized in terms of *quantitative linear-time temporal logic* (QLTL), a quantitative extension of linear-time temporal logic [13] that includes quantitative versions of the temporal operators and logic connectives. The QLTL formulas over a set $\Sigma$ of propositions are generated by the following grammar:

$$\varphi \quad ::= \quad D(r,c) \mid D(c,r) \mid \varphi\wedge\varphi \mid \varphi\vee\varphi \mid \quad \varphi \mid \Diamond\varphi \mid \Box\varphi$$

Here $r \in \Sigma$ is a proposition and $c \in \bigcup_{r\in\Sigma} X_r$ is a constant. We assume that, in a term of the form $D(r,c)$ or $D(c,r)$, we have $c \in X_r$. A formula $\varphi$ assigns a value $[\![\varphi]\!](\sigma) \in \mathbb{R}$ to each trace $\sigma \subseteq \mathcal{U}[\Sigma]^\omega$:

$$
\begin{array}{lcl}
[\![D(r,c)]\!](\sigma) & = & d_r(\sigma_0(r),c)\\
[\![D(c,r)]\!](\sigma) & = & d_r(c,\sigma_0(r))\\
[\![\varphi_1\wedge\varphi_2]\!](\sigma) & = & [\![\varphi_1]\!](\sigma) \sqcap [\![\varphi_2]\!](\sigma)\\
[\![\varphi_1\vee\varphi_2]\!](\sigma) & = & [\![\varphi_1]\!](\sigma) \sqcup [\![\varphi_2]\!](\sigma)\\
[\![\quad\varphi]\!](\sigma) & = & [\![\varphi]\!](\sigma^1)\\
[\![\Diamond\varphi]\!](\sigma) & = & \sup\{[\![\varphi]\!](\sigma^i) \mid i \ge 0\}\\
[\![\Box\varphi]\!](\sigma) & = & \inf\{[\![\varphi]\!](\sigma^i) \mid i \ge 0\}.
\end{array}
$$

A QLTL formula $\varphi$ assigns a real value $[\![\varphi]\!](s) \in \mathbb{R}$ to each state $s$ of a given MTS, by defining

$$[\![\varphi]\!](s) \;=\; \inf\{[\![\varphi]\!](\rho) \mid \rho \in \mathit{Tr}(s)\}.$$

We note that the above definition could also be phrased in terms of sup over all traces from $s$, rather than inf. However, as our setting is based on distances, the inf operator most closely corresponds to the universal quantification over all paths present in the classical definition of LTL semantics.

For *ops* $\subseteq \{\quad, \Diamond, \Box, D(c,r), D(r,c)\}$, we denote by QLTL$\setminus$ *ops* the set of formulas that do not employ the operators in *ops*.

Notice that QLTL is a proper extension to the fragment of LTL without the Until operator, in the following sense. Any Kripke structure $M$ has an obvious translation to an MTS $M'$ over $\mathbf{X}_{\mathbb{B}}$ (see Example 3). Moreover, any LTL formula $\varphi$ in positive normal form can be translated into a QLTL formula $\varphi'$ by replacing $r$ and $\neg r$ with $D(r,0)$ and $D(r,1)$, respectively. Then, $\varphi$ is true on a Kripke structure $M$ if and only if $\varphi'$ evaluates to 1 on $M'$.

## C. Logical Characterization of Linear Distances

Linear distances provide a bound for the difference in valuation of QLTL formulas. We begin by relating distances and logics over traces.

*Lemma 3:* *For all MTSs* $(S, \tau, \Sigma, [\cdot])$ *and all traces* $\sigma, \rho \in \mathcal{U}[\Sigma]^\omega$, *the following holds.*

*For all* $\varphi \in \text{QLTL} \setminus \{D(r,c)\}$ : $td(\sigma, \rho) \geq [\![\varphi]\!](\rho) - [\![\varphi]\!](\sigma)$.
*For all* $\varphi \in \text{QLTL} \setminus \{D(c,r)\}$ : $td(\sigma, \rho) \geq [\![\varphi]\!](\sigma) - [\![\varphi]\!](\rho)$.
*For all* $\varphi \in \text{QLTL}$ : $\overline{td}(\sigma, \rho) \geq |[\![\varphi]\!](\rho) - [\![\varphi]\!](\sigma)|$.

*Proof:* Let us consider the first assertion. We proceed by structural induction on $\varphi$. If $\varphi = D(c, r)$, using triangle inequality we get $[\![\varphi]\!](\rho) - [\![\varphi]\!](\sigma) = d(c, [\rho_0](r)) - d(c, [\sigma_0](r)) \leq d([\sigma_0](r), [\rho_0](r)) \leq pd(\sigma_0, \rho_0) \leq td(\sigma, \rho)$.

If $\varphi = \diamond\psi$, by inductive hypothesis we have that, for all $i \in \mathbb{N}$, $[\![\psi]\!](\rho^i) - [\![\psi]\!](\sigma^i) \leq td(\rho^i, \sigma^i)$. Then, by Lemma 1,

$$[\![\varphi]\!](\rho) - [\![\varphi]\!](\sigma) = \sup_{i \in \mathbb{N}} [\![\psi]\!](\rho^i) - \sup_{j \in \mathbb{N}} [\![\psi]\!](\sigma^j) \leq \sup_{i \in \mathbb{N}} td(\rho^i, \sigma^i) = td(\rho, \sigma).$$

Similar observations hold for the remaining cases.

The second assertion can be proved in a symmetrical fashion. The third assertion can be easily proved along similar lines. ■

The first result of the previous lemma is tight in two respects: both replacing $\text{QLTL} \setminus \{D(r,c)\}$ with QLTL and replacing $[\![\varphi]\!](\rho) - [\![\varphi]\!](\sigma)$ with $|[\![\varphi]\!](\rho) - [\![\varphi]\!](\sigma)|$ render the result false. The second assertion is tight in a similar sense. The following theorem uses the linear distances to provide the desired bounds for QLTL.

*Theorem 4:* *For all MTSs* $(S, \tau, \Sigma, [\cdot])$, *and all* $s, t \in S$, *the following holds.*
*For all* $\varphi \in \text{QLTL} \setminus \{D(r,c)\}$*:*

$$ld^{\mathrm{a}}(s,t) \geq [\![\varphi]\!](t) - [\![\varphi]\!](s) \text{ and } \overline{ld}^{\mathrm{a}}(s,t) \geq |[\![\varphi]\!](t) - [\![\varphi]\!](s)|.$$

*For all* $\varphi \in \text{QLTL}$*:*

$$ld^{\mathrm{s}}(s,t) \geq [\![\varphi]\!](t) - [\![\varphi]\!](s) \text{ and } \overline{ld}^{\mathrm{s}}(s,t) \geq |[\![\varphi]\!](t) - [\![\varphi]\!](s)|.$$

*Proof:* We first prove that $ld^{\mathrm{a}}(s,t) \geq [\![\varphi]\!](t) - [\![\varphi]\!](s)$.

$$\begin{aligned} ld^{\mathrm{a}}(s,t) &= \sup_{\sigma \in Tr(s)} \inf_{\rho \in Tr(t)} td(\sigma, \rho) \\ &\geq \sup_{\sigma \in Tr(s)} \inf_{\rho \in Tr(t)} ([\![\varphi]\!](\rho) - [\![\varphi]\!](\sigma)) \quad \text{by Lemma 3,} \\ &= \inf_{\rho \in Tr(t)} [\![\varphi]\!](\rho) - \inf_{\sigma \in Tr(s)} [\![\varphi]\!](\sigma) \\ &= [\![\varphi]\!](t) - [\![\varphi]\!](s). \end{aligned}$$

The result for $\overline{ld}^{\mathrm{a}}$ is an immediate consequence. The statements concerning $ld^{\mathrm{s}}$ and $\overline{ld}^{\mathrm{s}}$ follow in a similar way from Lemma 3. ■

The results for $ld^{\mathrm{s}}$ and $\overline{ld}^{\mathrm{s}}$ are the quantitative analogue of the standard connection between trace containment and trace equivalence, and LTL. For instance, the result about $ld^{\mathrm{s}}$ states that, if $ld^{\mathrm{s}}(s,t) = c$, then for every formula $\varphi \in \text{QLTL}$ and every trace $\sigma$ from $s$, there is a trace $\rho$ from $t$ such that $[\![\varphi]\!](\rho) \geq [\![\varphi]\!](\sigma) - c$.

We next show that, for finitely branching systems, QLTL provides a full logical characterization of the linear distances, meaning that the distinguishing power of the logic is exactly the same as the one of the distances. We start with a technical lemma. Given two traces $\sigma$ and $\rho$, and an integer $m$, let the *bounded distance* between $\sigma$ and $\rho$ be defined as $btd^m(\sigma, \rho) = \max_{0 \leq i \leq m} pd(\sigma_i, \rho_i)$. Clearly, $td(\sigma, \rho) = \lim_m btd^m(\sigma, \rho)$.

*Lemma 4:* *If the MTS* $M$ *is finitely branching, then for all traces* $\sigma$, *and* $t \in S$, *we have*

$$\sup_{m \in \mathbb{N}} \inf_{\rho \in Tr(t)} btd^m(\sigma, \rho) = \inf_{\rho \in Tr(t)} \sup_{m \in \mathbb{N}} btd^m(\sigma, \rho).$$

*Proof:* Since the l.h.s. is trivially smaller than or equal to the r.h.s., we are left to prove that $(l.h.s.) \geq (r.h.s.)$. Specifically, we prove that, for all $\epsilon > 0$, $(r.h.s.) \leq (l.h.s.) + \epsilon$. Fix $\epsilon > 0$. For all $m > 0$, there exists $\rho_m \in Tr(t)$ such that

$$btd^m(\sigma, \rho_m) \leq \inf_{\rho \in Tr(t)} btd^m(\sigma, \rho) + \epsilon.$$

For all $m \geq 0$, let $\gamma_m$ be the prefix of $\rho_m$ up to the $m+1$-th valuation. The set $\{\gamma_m \mid m \geq 0\}$ can be arranged into a tree that is a subtree of the unrolling of $t$. Since this tree contains infinitely many nodes and is finitely branching, by König's lemma it must contain an infinite trace $\rho^* \in Tr(t)$. The trace $\rho^*$ has infinitely many prefixes in $\{\gamma_m \mid m \geq 0\}$. Therefore, there is an increasing sequence of indices $(i_m)_{m>0}$ such that, for all $m \geq 0$, $\gamma_{i_m}$ is a prefix of $\rho^*$. It follows that

$$\begin{aligned} (r.h.s.) \leq td(\sigma, \rho^*) &= \lim_m btd^m(\sigma, \rho^*) \\ &= \lim_m btd^{i_m}(\sigma, \rho^*) \\ &\leq \lim_m btd^{i_m}(\sigma, \gamma_{i_m}) \\ &= \lim_m btd^{i_m}(\sigma, \rho_{i_m}) \\ &\leq \lim_m \inf_{\rho \in Tr(t)} btd^{i_m}(\sigma, \rho) + \epsilon \\ &= (l.h.s.) + \epsilon. \end{aligned}$$

■

The following theorem identifies the fragments of the logics that suffice for characterizing each linear distance. In particular, the theorem shows that the operators $\diamond$ and $\Box$ are never needed. Together with Theorem 4, this result constitutes a full characterization of linear distances in terms of QLTL.

*Theorem 5:* *If an MTS* $M = (S, \tau, \Sigma, [\cdot])$ *is finitely branching, then we have for all* $s, t \in S$ *that*

$$\begin{aligned} ld^{\mathrm{a}}(s,t) &= \sup_{\varphi \in \text{QLTL} \setminus \{D(r,c), \diamond, \Box\}} [\![\varphi]\!](t) - [\![\varphi]\!](s) \\ \overline{ld}^{\mathrm{a}}(s,t) &= \sup_{\varphi \in \text{QLTL} \setminus \{D(r,c), \diamond, \Box\}} |[\![\varphi]\!](t) - [\![\varphi]\!](s)| \\ ld^{\mathrm{s}}(s,t) &= \sup_{\varphi \in \text{QLTL} \setminus \{\diamond, \Box\}} [\![\varphi]\!](t) - [\![\varphi]\!](s) \\ \overline{ld}^{\mathrm{s}}(s,t) &= \sup_{\varphi \in \text{QLTL} \setminus \{\diamond, \Box\}} |[\![\varphi]\!](t) - [\![\varphi]\!](s)|. \end{aligned}$$

*Proof:* By Theorem 4, we only need to prove the "$\leq$" part of the equalities. We first prove the statement involving $ld^{\mathrm{a}}$. For the sake of simplicity, assume $\Sigma = \{r\}$. Let $ld^{\mathrm{a}}(s,t) = x$, we show that for all $\epsilon > 0$ there is a formula $\varphi$ such that $[\![\varphi]\!](t) - [\![\varphi]\!](s) > x - \epsilon$. Let $\sigma^* \in Tr(s)$ be a trace such that $\inf_{\rho \in Tr(t)} td(\sigma^*, \rho) > x - \epsilon$. For all $m \geq 0$, we set

$$\varphi_m = \bigvee_{0 \leq i \leq m} {}^{i}D([\sigma^*_i](r), r),$$

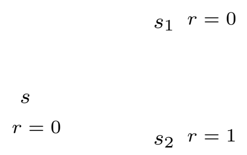



Fig. 4. An MTS exhibiting the language $0\{0,1\}^\omega$; the single proposition is evaluated in the metric space $\mathbf{X}_\mathbb{B}$.

where $\ ^i$ stands for $i$ repetitions of the operator . Intuitively, when formula $\varphi_m$ is evaluated on a trace $\sigma'$, it measures the asymmetric distance between $\sigma'$ and $\sigma^*$, up to the $m$-th step. Obviously, we have $[\![\varphi_m]\!](s) = 0$ for all $m \geq 0$. Then, the value of $\varphi_m$ on a state $s'$ measures the distance between $\sigma^*$ and the trace in $Tr(s')$ which is closest to it. For all $t \in S$, it holds that

$$\begin{aligned} \sup_m [\![\varphi_m]\!](t) &= \lim_m [\![\varphi_m]\!](t) \\ &= \lim_m \inf_{\rho \in Tr(t)} \max_{0 \leq i \leq m} D([\sigma_i^*](r), [\rho_i](r)) \end{aligned}$$

since $[\![\varphi_{m+1}]\!](t) \geq [\![\varphi_m]\!](t)$

$$\begin{aligned} &= \lim_m \inf_{\rho \in Tr(t)} btd^m(\sigma^*, \rho) \\ &= \inf_{\rho \in Tr(t)} td(\sigma^*, \rho) \qquad \text{by Lemma 4} \\ &> x - \epsilon. \end{aligned}$$

Consequently,

$$\begin{aligned} \sup_{\varphi \in \text{QLTL} \setminus \{D(r,c), \diamond, \square\}} [\![\varphi]\!](t) - [\![\varphi]\!](s) &\geq \sup_{m \in \mathbb{N}} [\![\varphi_m]\!](t) - [\![\varphi_m]\!](s) \\ &= \sup_{m \in \mathbb{N}} [\![\varphi_m]\!](t) - 0 \\ &> x - \epsilon. \end{aligned}$$

The statement about $\overline{ld}^{\mathrm{a}}$ is an easy consequence: Assume first that $\overline{ld}^{\mathrm{a}}(s,t) = ld^{\mathrm{a}}(s,t)$. Then,

$$\begin{aligned} \overline{ld}^{\mathrm{a}}(s,t) &= \sup_{\varphi \in \text{QLTL} \setminus \{D(r,c), \diamond, \square\}} [\![\varphi]\!](s) - [\![\varphi]\!](t) \\ &\leq \sup_{\varphi \in \text{QLTL} \setminus \{D(r,c), \diamond, \square\}} |[\![\varphi]\!](s) - [\![\varphi]\!](t)|. \end{aligned}$$

If instead $\overline{ld}^{\mathrm{a}}(s,t) = ld^{\mathrm{a}}(t,s)$, we have

$$\begin{aligned} \overline{ld}^{\mathrm{a}}(s,t) &= \sup_{\varphi \in \text{QLTL} \setminus \{D(r,c), \diamond, \square\}} [\![\varphi]\!](t) - [\![\varphi]\!](s) \\ &\leq \sup_{\varphi \in \text{QLTL} \setminus \{D(r,c), \diamond, \square\}} |[\![\varphi]\!](s) - [\![\varphi]\!](t)|. \end{aligned}$$

We now consider the statement about $ld^{\mathrm{s}}$. The proof proceeds similarly to the one involving $ld^{\mathrm{a}}$, using as distinguishing formula the following.

$$\varphi_m = \bigvee_{0 \leq i \leq m} \ ^i D([\sigma_i^*](r), r) \vee \ ^i D(r, [\sigma_i^*](r)).$$

Finally, the statement involving $\overline{ld}^{\mathrm{s}}$ can be easily obtained from the one involving $ld^{\mathrm{s}}$ and from the fact that $\overline{ld}^{\mathrm{s}}(s,t) = ld^{\mathrm{s}}(s,t) \sqcup ld^{\mathrm{s}}(t,s)$. ■

The next result shows that Theorem 5 does not hold for non-finite-branching systems.

*Theorem 6:* *There is an infinitely branching MTS such that*

$$ld^{\mathrm{s}}(s,t) > \sup_{\varphi \in \text{QLTL}} [\![\varphi]\!](s) - [\![\varphi]\!](t).$$

*Proof:* Consider the system in Figure 4, where $\Sigma = \{r\}$. Informally, $Tr(s) = 0\{0,1\}^\omega$. Let $\sigma$ be a trace such that $\{\sigma\}$ is not a regular language over the alphabet $\{0,1\}$ (it would be sufficient for $\sigma$ to be not star-free regular). For instance, let $\sigma = 01\,001\,0001\ldots$. Consider a second system, containing a state $t$ such that $Tr(t) = Tr(s) \setminus \{\sigma\}$. Notice that, in order to have such a set of traces, $t$ must be infinitely branching, since if a finitely branching tree contains all prefixes of an infinite path, it must also contain the path itself. We have $ld^{\mathrm{s}}(s,t) = 1$. We know that ordinary LTL cannot distinguish $s$ from $t$, otherwise there would be a formula $\psi \in$ LTL such that the set of traces that satisfy $\psi$ is $\{\sigma\}$. This is impossible since LTL can only express star-free regular languages. As observed in Section III-B, if all propositions are evaluated on $\mathbf{X}_\mathbb{B}$, an MTS is equivalent to a Kripke structure, and QLTL is equivalent to LTL. Thus, QLTL is also unable to distinguish $s$ from $t$. ■

Above, we have provided a logical characterization for the linear distances in terms of a logic that contains a potentially uncountable set of constants: in general, we need one constant for each element of a metric space corresponding to a proposition. However, for separable MTSs we can provide a characterization in terms of logics with countably many symbols. First, we prove that small changes in the value of the constants cause small changes in the value of the formulas. The result follows by a straightforward structural induction.

*Theorem 7:* *Consider a* QLTL *formula $\varphi$ containing the constants $c_1, \ldots, c_n$, belonging respectively to the metric spaces $(X_1, d_1), \ldots, (X_n, d_n)$. Let $\psi$ be the result of replacing in $\varphi$ each $c_i$ with $c_i'$, for $1 \leq i \leq n$, and let $\delta = \max_{i=1}^n (d_i(c_i, c_i') \sqcup d_i(c_i', c_i))$ be the maximal distance between the new and old values of each constant. Then, for all $s \in S$, we have $|[\![\varphi]\!](s) - [\![\psi]\!](s)| \leq \delta$.*

From the above result, it follows that if an MTS is separable, we can obtain a logical characterization of the linear distances in terms of logics that consist only of countably many symbols. The idea, essentially, is to replace each constant with a nearby element of a countable base in the formulas used to characterize the distances.

*Theorem 8:* *If an MTS $M = (S, \tau, \Sigma, [\cdot])$ is both finitely branching and separable, then the characterizations provided by Theorem 5 hold also when we restrict the formulas of* QLTL *to those containing only constants from the countable set $\bigcup_{r \in \Sigma} \mathcal{B}_r$, where $\mathcal{B}_r$ is a countable basis for the metric space $(X_r, d_r)$, for each $r \in \Sigma$.*

*Proof:* The result follows immediately from the observation that by Theorem 7 the value of a formula, at every state, can be approximated arbitrarily well by the value of a formula containing only constants that belong to the countable bases of the metric spaces. ■

### D. A Note on Algorithmic Complexity

The following section describes an algorithm that takes as input a finite MTS $M$ and computes the value of a linear distance between all pairs of states. To discuss its complexity, we need to fix a finite representation for the input data. Considering that all the linear distances have as starting point the propositional distance $pd$, it is sufficient to provide as input the $|S| \times |S|$ matrix $A = (a_{s,t})_{s,t \in S}$, where $a_{s,t} = pd(s,t)$.

We assume that the values $pd(s,t)$ are rational numbers encoded in fixed-precision binary representation; we denote by $|x|_b$ the number of bits in the encoding of the rational number $x$. We define the size of a finite MTS $M = (S, \tau, \Sigma, [\cdot])$ by $|M| = \sum_{s,t \in S} |pd(s,t)|_b$. The size of an MTS is thus quadratic in $|S|$. We further assume that any arithmetic operation between rationals can be carried out in constant time.

### E. Computing the Linear Distance

Given as inputs a finite MTS $M = (S, \tau, \Sigma, [\cdot])$, and $x \in \{\mathrm{a}, \mathrm{s}\}$, we wish to compute $ld^x(s_0, t_0)$, for all $s_0, t_0 \in S$.

We describe the computation of $ld^{\mathrm{a}}$, as the computation of $ld^{\mathrm{s}}$ is analogous. We can read the definition of $ld^{\mathrm{a}}$ as a two-player game. Player 1 chooses a path $\pi = s_0 s_1 s_2 \cdots$ from $s_0$; Player 2 chooses a path $\pi' = t_0 t_1 t_2 \cdots$ from $t_0$; the goal of Player 1 (resp. Player 2) is to maximize (resp. minimize) $\sup_k pd(\pi_k, \pi'_k)$. The game is played with partial information: after $s_0 \cdots s_n$, Player 1 must choose $s_{n+1}$ without knowledge[2] of $t_0 \cdots t_n$. Such a game can be solved via a variation of the subset construction [15]. The key idea is to associate with each final state $s_n$ of a finite path $s_0 s_1 \cdots s_n$ chosen by Player 1, all final states $t_n$ of finite paths $t_0 t_1 \cdots t_n$ chosen by Player 2, each labeled by the distance $v(s_0 \cdots s_n, t_0 \cdots t_n) = \max_{0 \le k \le n} pd(s_k, t_k)$.

Formally, from $M$, we construct another MTS $M' = (S', \tau', \{r\}, [\cdot]')$, having set of states $S' = S \times 2^{S \times \mathbb{D}}$. Here, $\mathbb{D} = \{pd(s,t) \mid s,t \in S\}$, so that $|\mathbb{D}| \le |S|^2$. The transition relation $\tau'$ consists of all pairs $(\langle s, C\rangle, \langle s', C'\rangle)$ such that $s' \in \tau(s)$ and $C' = \{\langle t', v'\rangle \mid \exists \langle t, v\rangle \in C \,.\, t' \in \tau(t) \wedge v' = v \sqcup pd(s', t')\}$. Note that only Player 1 has a choice of moves in this game, since the moves of Player 2 are accounted for by the subset construction. Finally, the proposition $r$ is interpreted over $X_r = (\mathbb{D}, d_{\mathbb{R}})$, and the interpretation $[\cdot]'$ is given by $[\langle s, C\rangle]'(r) = \min\{v \mid \langle t, v\rangle \in C\}$, so that $r$ indicates the minimum distance achievable by Player 2 while trying to match a path to $\langle s, C\rangle$ chosen by Player 1.

The goal of the game, for Player 1, consists in reaching a state of $M'$ with the highest possible value of $r$. Let $r_{\max} = \max \mathbb{D}$, for all $s,t \in S$, we have $ld^{\mathrm{a}}(s,t) = r_{\max} - [\![\Box D(r_{\max}, r)]\!](\langle s, \{\langle t, pd(s,t)\rangle\}\rangle)$, where the right-hand side is to be computed on $M'$. This expression can be evaluated by a depth-first traversal of the state space of $M'$, noting that no state of $M'$ needs to be visited twice, as repeated visits cannot modify the value of $\Box D(r_{\max}, r)$ (see Lemma 3 from [3]). This leads to the following complexity result.

*Theorem 9:* *For all $x \in \{\mathrm{a}, \mathrm{s}\}$, the following assertions hold:*

1) *Computing $ld^x$ for an MTS $M$ is PSPACE-complete in $|M|$.*
2) *Computing $ld^x$ for a deterministic MTS $M$ is PSPACE-complete in $|M|$.*
3) *Computing $ld^x$ for a boolean, deterministic MTS $M$ is in time $O(|M|^4)$.*

*Proof:* For Part 1, the upper complexity bound comes from the above algorithm, noticing that the subset construction can be done on the fly; the lower bound comes from a reduction from the corresponding result for trace inclusion [16].

Part 2 states that, unlike in the boolean case, the problem remains PSPACE-complete even for deterministic MTSs. This result is proved by an nlogspace reduction from the problem of computing trace inclusion for nondeterministic boolean systems.

Consider an MTS $M_b = (S, \tau, \Sigma, [\cdot])$ where all the propositions in $\Sigma$ take value in $\mathbf{X}_{\mathbb{B}}$; hence, $M_b$ is a transition system with states that assign boolean values to propositions. Given $s,t \in S$, the problem of deciding trace inclusion between $s$ and $t$ is PSPACE-complete [16]. We provide a nlogspace reduction from this problem to the problem of computing the linear distance $ld^{\mathrm{s}}(s,t)$ in a *deterministic* MTS. Note that, for $M_b$, the distance matrix $A$ is of the same size as the representation of $\tau$ via the adjacency matrix $S \times S \mapsto \{0,1\}$.

We build a deterministic MTS $M' = (S, \tau, \Sigma, [\cdot]')$, where all propositions $r \in \Sigma$ are interpreted in the metric space $([0,n], d_{\mathbb{R}})$, and $[\cdot]'$ is defined as follows. Let the elements of $S$ be numbered as $s_0, \ldots, s_n$. For all $i = 0 \ldots n$ and $r \in \Sigma$, we set

$$[s_i]'(r) = \begin{cases} i & \text{if } [s_i](r) = 0 \\ 4n - i & \text{if } [s_i](r) = 1 \end{cases}$$

By construction, $M'$ is deterministic and its size is polynomial in the size of $M$, as $\lceil \log(n+1) \rceil + 2$ bits are sufficient to represent the value of a proposition in a state of $M'$, as well as the difference in value between two states. Finally, the proof is completed by the observation that $s \sqsubseteq_{tr} t$ in $M$ if and only if $ld^{\mathrm{s}}(s,t) \le n$ in $M'$.

Part 3 is a consequence of Theorems 16 and 17. ■

### F. Discussion

In Definition 10, we could have defined the propositional distance between two states using the $L_2$ norm, via $pd(u,v) = \left(\sum_{r \in \Sigma} d(u(r), v(r))^2\right)^{1/2}$ (or in general using the $L_n$ norm, for $n > 0$). The reason why in Definition 10 we chose the $L_\infty$ norm is that this definition leads to a logical characterization of the distances, since the $\max$ in the $L_\infty$ norm corresponds to the $\vee$ of the logics. It is easy to see that, aside from the logical characterizations, the results of the paper would hold if we replaced in Definition 10 the $L_\infty$ norm with $L_n$, for any $n > 0$.

## IV. Branching Distances and Logics

### A. Branching Distances

*Definition 13:* **(branching distances)** For $x \in \{\mathrm{Aa}, \mathrm{As}, \mathrm{Sa}, \mathrm{Ss}\}$, consider the four operators $H^x : (S^2 \to$

[2]Indeed, if the game were played with total information, we would obtain the branching distances of the next section.

$\mathbb{R}) \to (S^2 \to \mathbb{R})$ defined as follows, for $d : S^2 \to \mathbb{R}$:

$$\begin{aligned}
H^{\mathrm{Aa}}(d)(s,t) &= pd(s,t) && \sqcup \sup_{s' \in \tau(s)} \inf_{t' \in \tau(t)} d(s',t') \\
H^{\mathrm{As}}(d)(s,t) &= \overline{pd}(s,t) && \sqcup \sup_{s' \in \tau(s)} \inf_{t' \in \tau(t)} d(s',t') \\
H^{\mathrm{Sa}}(d)(s,t) &= pd(s,t) && \sqcup \sup_{s' \in \tau(s)} \inf_{t' \in \tau(t)} d(s',t') \\
& && \sqcup \sup_{t' \in \tau(t)} \inf_{s' \in \tau(s)} d(s',t') \\
H^{\mathrm{Ss}}(d)(s,t) &= \overline{pd}(s,t) && \sqcup \sup_{s' \in \tau(s)} \inf_{t' \in \tau(t)} d(s',t') \\
& && \sqcup \sup_{t' \in \tau(t)} \inf_{s' \in \tau(s)} d(s',t').
\end{aligned}$$

For $x \in \{\mathrm{Aa}, \mathrm{As}, \mathrm{Sa}, \mathrm{Ss}\}$, we define the *branching distance* $bd^x$ as the least fixpoint of the operator $H^x$. ■

The functions $bd^{\mathrm{Aa}}$, $bd^{\mathrm{As}}$, and $bd^{\mathrm{Sa}}$ are directed metrics, while $bd^{\mathrm{Ss}}$, $\overline{bd}^{\mathrm{Aa}}$, $\overline{bd}^{\mathrm{As}}$, and $\overline{bd}^{\mathrm{Sa}}$ are undirected metrics.

*Example 8:* Consider the MTS in Figure 1 once more. We have for instance, $bd^{\mathrm{As}}(s_1,t_1) = bd^{\mathrm{As}}(s_3,t_3) \sqcup bd^{\mathrm{As}}(s_4,t_3) = 0.1 \sqcup 0.2 = 0.2$: both transitions in $s_1$ need to be matched by transitions from $t_1$. Similarly, $bd^{\mathrm{As}}(s_1,t_2) = bd^{\mathrm{As}}(s_3,t_4) \sqcup bd^{\mathrm{As}}(s_4,t_4) = 0.6 \sqcup 0.3 = 0.6$. Thus, $bd^{\mathrm{As}}(s_0,t_0) = bd^{\mathrm{As}}(s_1,t_1) \sqcap bd^{\mathrm{As}}(s_1,t_2) = 0.3 \sqcap 0.6 = 0.3$: we match $s_0 \to s_1$ by $t_0 \to t_1$, because state $t_1$ has the smallest branching distance to $s_1$. ■

The distance $bd^{\mathrm{Ss}}$ is a quantitative generalization of bisimulation, and it essentially coincides with the metrics of [8], [17], [4]; as it is already symmetrical, we have $\overline{bd}^{\mathrm{Ss}} = bd^{\mathrm{Ss}}$. Similarly, the distance $bd^{\mathrm{As}}$ generalizes simulation, and $\overline{bd}^{\mathrm{As}}$ generalizes mutual simulation.

*Theorem 10: For all finitely branching MTSs $(S, \tau, \Sigma, [\cdot])$ such that $d_r$ is a proper metric for all $r \in \Sigma$, we have $\preceq_{sim} = \mathrm{Zero}(bd^{\mathrm{As}})$ and $\approx_{bis} = \mathrm{Zero}(bd^{\mathrm{Ss}})$.*

The necessity for the finitely branching condition is again shown by the MTS in Figure 2, where we have $bd^{\mathrm{As}}(s_0,t_0) = 0$, but $s_0 \not\preceq_{sim} t_0$.

The distances $bd^{\mathrm{Aa}}$ and $bd^{\mathrm{Sa}}$ correspond to quantitative notions of simulation and bisimulation with respect to the asymmetrical propositional distance $pd$; these distances are not symmetrical, and we indicate their symmetrical versions by $\overline{bd}^{\mathrm{Aa}}$ and $\overline{bd}^{\mathrm{Sa}}$. Just as in the boolean case mutual similarity is not equivalent to bisimulation, so in our quantitative setting $\overline{bd}^{\mathrm{As}}$ can be strictly smaller than $bd^{\mathrm{Ss}}$, and $\overline{bd}^{\mathrm{Aa}}$ can be strictly smaller than $bd^{\mathrm{Sa}}$.

*Theorem 11: The relations in Figure 6(b) hold for all MTS and no other inequalities on these relations hold on all MTSs.*

*Proof:* The inequalities $bd^{\mathrm{Aa}} \leq bd^{\mathrm{Sa}} \leq bd^{\mathrm{Ss}}$ and $bd^{\mathrm{Aa}} \leq bd^{\mathrm{As}} \leq bd^{\mathrm{Ss}}$ shown in the figure are immediate. Consider the MTS in Figure 3 again. In this MTS, we have $ld^{\mathrm{a}} = bd^{\mathrm{Aa}}$, $ld^{\mathrm{s}} = \overline{bd}^{\mathrm{As}}$, $\overline{ld}^{\mathrm{a}} = \overline{bd}^{\mathrm{Sa}}$, $\overline{ld}^{\mathrm{s}} = bd^{\mathrm{Ss}}$ Hence, the results for the linear distances (see Theorem 2) show that $bd^{\mathrm{Aa}} \neq \overline{bd}^{\mathrm{As}}$, $bd^{\mathrm{Aa}} \neq \overline{bd}^{\mathrm{Sa}}$, $bd^{\mathrm{As}} \neq bd^{\mathrm{Ss}}$, $bd^{\mathrm{Sa}} \neq bd^{\mathrm{Ss}}$, and neither $bd^{\mathrm{As}} \leq bd^{\mathrm{Sa}}$ nor $bd^{\mathrm{As}} \geq bd^{\mathrm{Sa}}$. ■

The branching distances, like the linear ones, are robust with respect to perturbations in the state valuations: small changes in the propositional valuations cause small changes in the distances. To state the theorem, given a state valuation $f : S \to \mathcal{U}[\Sigma]$, $x \in \{\mathrm{Aa}, \mathrm{As}, \mathrm{Sa}, \mathrm{Ss}\}$, we write $bd_f^x$ for the distances defined as in Definition 13, using $f$ as the state valuation.

*Theorem 12:* **(branching distance robustness)** *For all $x \in \{\mathrm{As}, \mathrm{Sa}, \mathrm{Ss}\}$, all propositional valuations $[\cdot]_1, [\cdot]_2$, and all $s, t \in S$, we have*

$$bd_{[\cdot]_1}^{Aa}(s,t) - bd_{[\cdot]_2}^{Aa}(s,t) \leq d([\cdot]_1, [\cdot]_2) + d([\cdot]_2, [\cdot]_1)$$
$$|bd_{[\cdot]_1}^{x}(s,t) - bd_{[\cdot]_2}^{x}(s,t)| \leq 2 \cdot \overline{d}([\cdot]_1, [\cdot]_2).$$

### *B. Quantitative $\mu$-Calculus*

We define quantitative $\mu$-calculus after [5], [4]. Given a set of variables $V$ and a set of propositions $\Sigma$, the formulas of the *quantitative $\mu$-calculus* are generated by the grammar:

$$\varphi ::= D(r,c) \mid D(c,r) \mid x \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists \quad \varphi \mid \forall \quad \varphi \mid \mu x \,.\, \varphi \mid \nu x \,.\, \varphi$$

for propositions $r \in \Sigma$, variables $x \in V$, and constants $c \in \bigcup_{r \in \Sigma} X_r$. We assume that, in a term of the form $D(r,c)$ or $D(c,r)$, we have $c \in X_r$. Denoting by $\mathcal{F} = (S \to \mathbb{R})$, a (variable) interpretation is a function $\mathcal{E} : V \to \mathcal{F}$. Given an interpretation $\mathcal{E}$, a variable $x \in V$ and a function $f \in \mathcal{F}$, we denote by $\mathcal{E}[x := f]$ the interpretation $\mathcal{E}'$ such that $\mathcal{E}'(x) = f$ and, for all $y \neq x$, $\mathcal{E}'(y) = \mathcal{E}(y)$. Given an MTS and an interpretation $\mathcal{E}$, every formula $\varphi$ of the quantitative $\mu$-calculus defines a valuation $[\![\varphi]\!]_{\mathcal{E}} : S \to \mathbb{R}$:

$$\begin{aligned}
[\![D(r,c)]\!]_{\mathcal{E}}(s) &= d([s](r), c) \\
[\![D(c,r)]\!]_{\mathcal{E}}(s) &= d(c, [s](r)) \\
[\![x]\!]_{\mathcal{E}} &= \mathcal{E}(x) \\
[\![\varphi_1 \wedge \varphi_2]\!]_{\mathcal{E}} &= [\![\varphi_1]\!]_{\mathcal{E}} \sqcap [\![\varphi_2]\!]_{\mathcal{E}} \\
[\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{E}} &= [\![\varphi_1]\!]_{\mathcal{E}} \sqcup [\![\varphi_2]\!]_{\mathcal{E}} \\
[\![\exists \quad \varphi]\!]_{\mathcal{E}}(s) &= \sup\nolimits_{s' \in \tau(s)} [\![\varphi]\!]_{\mathcal{E}}(s') \\
[\![\forall \quad \varphi]\!]_{\mathcal{E}}(s) &= \inf\nolimits_{s' \in \tau(s)} [\![\varphi]\!]_{\mathcal{E}}(s') \\
[\![\mu x \,.\, \varphi]\!]_{\mathcal{E}} &= \inf\{f \in \mathcal{F} \mid f = [\![\varphi]\!]_{\mathcal{E}[x:=f]}\} \\
[\![\nu x \,.\, \varphi]\!]_{\mathcal{E}} &= \sup\{f \in \mathcal{F} \mid f = [\![\varphi]\!]_{\mathcal{E}[x:=f]}\}.
\end{aligned}$$

The existence of the required fixpoints is guaranteed by the monotonicity and continuity of all operators. A variable $x$ is *bound* in $\varphi$ if it is in the scope of a quantifier $\mu x$ or $\nu x$; otherwise, it is called *free*. A formula is *closed* if all variables are bound. If $\varphi$ is closed, we write $[\![\varphi]\!]$ for $[\![\varphi]\!]_{\mathcal{E}}$. We call QMU the set of quantitative $\mu$-calculus formulas and denote by CLQMU the subset of QMU containing only closed formulas. For *ops* $\subseteq \{D(c,r), D(r,c), \exists \ , \forall \ , \mu, \nu\}$, we denote by QMU$\backslash$*ops* and CLQMU$\backslash$*ops* the respective subsets of formulas that do not employ operators in *ops*. Notice that, on boolean systems, the semantics of the quantitative $\mu$-calculus coincides with the classical $\mu$-calculus semantics.

### *C. Logical Characterizations of Branching Distances*

In the following theorem, we write $\varphi(x_1, \ldots, x_n)$ to signify that the free variables in $\varphi$ are among $x_1, \ldots, x_n$.

*Lemma 5: For all finitely branching MTSs $(S, \tau, \Sigma, [\cdot])$ and all variable interpretations $\mathcal{E}$, the following holds.*

1) *For all* $\varphi(x_1,\ldots,x_n) \in$ QMU$\setminus\{\exists\ \ , D(r,c)\}$ *and for all* $f_1,\ldots,f_n \in \mathcal{F}$*, if for all* $s,t \in S$ *and all* $i = 1,\ldots,n$*,* $f_i(t) - f_i(s) \le bd^{\mathrm{Aa}}(s,t)$*, then, for all* $s,t \in S$*,*
$$[\![\varphi]\!]_{\mathcal{E}[x_i:=f_i]}(t) - [\![\varphi]\!]_{\mathcal{E}[x_i:=f_i]}(s) \le bd^{\mathrm{Aa}}(s,t).$$
2) *For all* $\varphi(x_1,\ldots,x_n) \in$ QMU $\setminus \{\exists\ \ \}$ *and for all* $f_1,\ldots,f_n \in \mathcal{F}$*, if for all* $s,t \in S$ *and all* $i = 1,\ldots,n$*,* $f_i(t) - f_i(s) \le bd^{\mathrm{As}}(s,t)$*, then, for all* $s,t \in S$*,*
$$[\![\varphi]\!]_{\mathcal{E}[x_i:=f_i]}(t) - [\![\varphi]\!]_{\mathcal{E}[x_i:=f_i]}(s) \le bd^{\mathrm{As}}(s,t).$$
3) *For all* $\varphi(x_1,\ldots,x_n) \in$ QMU $\setminus \{D(r,c)\}$ *and for all* $f_1,\ldots,f_n \in \mathcal{F}$*, if for all* $s,t \in S$ *and all* $i = 1,\ldots,n$*,* $f_i(t) - f_i(s) \le bd^{\mathrm{Sa}}(s,t)$*, then, for all* $s,t \in S$*,*
$$[\![\varphi]\!]_{\mathcal{E}[x_i:=f_i]}(t) - [\![\varphi]\!]_{\mathcal{E}[x_i:=f_i]}(s) \le bd^{\mathrm{Sa}}(s,t).$$
4) *For all* $\varphi(x_1,\ldots,x_n) \in$ QMU *and for all* $f_1,\ldots,f_n \in \mathcal{F}$*, if for all* $s,t \in S$ *and all* $i = 1,\ldots,n$*,* $|f_i(t) - f_i(s)| \le bd^{\mathrm{Ss}}(s,t)$*, then, for all* $s,t \in S$*,*
$$|[\![\varphi]\!]_{\mathcal{E}[x_i:=f_i]}(t) - [\![\varphi]\!]_{\mathcal{E}[x_i:=f_i]}(s)| \le bd^{\mathrm{Ss}}(s,t).$$

*Proof:* We prove statements 1 and 3; the other two statements can be proved in similar fashion.

*Statement 1:* We prove the result concerning $bd^{\mathrm{Aa}}$ by structural induction on the formula. For $\varphi = D(c,r)$, we obtain by triangle inequality $[\![\varphi]\!](t) - [\![\varphi]\!](s) = d(c,[t](r)) - d(c,[s](r)) \le d([s](r),[t](r)) \le pd(s,t) \le bd^{\mathrm{Aa}}(s,t)$. The cases $\varphi = x$, $\varphi = \varphi_1 \wedge \varphi_2$ and $\varphi = \varphi_1 \vee \varphi_2$ are also trivial.

Consider the case $\varphi = \forall\ \ \psi$. For ease of notation, in this part of the proof we write $[\![\cdot]\!]$ for $[\![\cdot]\!]_{\mathcal{E}[x_i:=f_i]}$, since the variable interpretation is not the issue here. Recall that, for all $t \in S$, we have by definition $[\![\varphi]\!](t) = \inf_{t' \in \tau(t)} [\![\psi]\!](t')$. By inductive hypothesis, for all $s',t' \in S$, $[\![\psi]\!](t') - [\![\psi]\!](s') \le bd^{\mathrm{Aa}}(s',t')$. We have

$$\begin{aligned}[\![\varphi]\!](t) - [\![\varphi]\!](s) &= \inf_{t'\in\tau(t)} [\![\psi]\!](t') - \inf_{s'\in\tau(s)} [\![\psi]\!](s') \\ &= \sup_{s'\in\tau(s)} \inf_{t'\in\tau(t)} \big([\![\psi]\!](t') - [\![\psi]\!](s')\big) \\ &\le \sup_{s'\in\tau(s)} \inf_{t'\in\tau(t)} bd^{\mathrm{Aa}}(s',t')\end{aligned}$$

by induction

$$\le bd^{\mathrm{Aa}}(s,t).$$

This concludes this case.

If $\varphi = \mu y \,.\, \psi$, then $[\![\varphi]\!] = \lim_n g_n$, where $g_0(s) = 0$ for all $s \in S$, and $g_{n+1} = [\![\psi]\!]_{\mathcal{E}[y:=g_n]}$. This is a consequence of the fact that, when the MTS is finitely branching, all operators of the $\mu$-calculus are continuous: that is, for each operator $F \in \{\wedge, \vee, \exists\ \ , \forall\ \ \}$ and each sequence $\{g_n\}_{n\ge 0}$ of functions $S^2 \to \mathbb{R}$, we have $F(\lim_n g_n) = \lim_n F(g_n)$. Since $g_0(t) - g_0(s) = 0 \le bd^{\mathrm{Aa}}(s,t)$, by inductive hypothesis we obtain that, for all $n \in \mathbb{N}$, $g_n(t) - g_n(s) \le bd^{\mathrm{Aa}}(s,t)$, and thus the thesis. If $\varphi = \nu y \,.\, \psi$, we proceed similarly, except that the initial function $g_0$ must assign to each state a value which is greater than any possible value of formula $\psi$ on the current MTS. Such a value can easily be found, since all metric spaces giving value to propositions are bounded. Namely, any real number greater than the greatest diameter of those metric spaces can be used as value for $g_0(s)$, for all $s \in S$.

*Statement 3:* The cases $\varphi = D(c,r)$, $\varphi = x$, $\varphi = \psi_1 \wedge \psi_2$ and $\varphi = \psi_1 \vee \psi_2$ are trivial, while the proofs for $\varphi = \forall\ \ \psi$, $\varphi = \mu y \,.\, \psi$ and $\varphi = \nu y \,.\, \psi$ are similar to the ones of Statement 1.

Let $\varphi = \exists\ \ \psi$. For ease of notation, we again write $[\![\cdot]\!]$ for $[\![\cdot]\!]_{\mathcal{E}[x_i:=f_i]}$. By inductive hypothesis, for all $s',t' \in S$, $[\![\psi]\!](t') - [\![\psi]\!](s') \le bd^{\mathrm{Sa}}(s',t')$.

Similarly to Statement 1, we have

$$\begin{aligned}[\![\varphi]\!](t) - [\![\varphi]\!](s) &= \sup_{t'\in\tau(t)} [\![\psi]\!](t') - \sup_{s'\in\tau(s)} [\![\psi]\!](s') \\ &= \sup_{t'\in\tau(t)} \inf_{s'\in\tau(s)} \big([\![\psi]\!](t') - [\![\psi]\!](s')\big) \\ &\le \sup_{t'\in\tau(t)} \inf_{s'\in\tau(s)} bd^{\mathrm{Sa}}(s',t')\end{aligned}$$

by induction

$$\le bd^{\mathrm{Sa}}(s,t),$$

leading to the desired result. ∎

From the preceding lemma, we immediately obtain a theorem stating that the branching distances provide bounds for the corresponding fragments of the $\mu$-calculus. The statement for $bd^{\mathrm{Ss}}$ is very similar to a result in [8].

*Theorem 13:* *For all finitely branching MTSs* $(S,\tau,\Sigma,[\cdot])$*, states* $s,t \in S$*, we have*

$$\begin{array}{ll}\forall \varphi \in \text{CLQMU}\setminus\{\exists\ \ , D(r,c)\} & bd^{\mathrm{Aa}}(s,t) \ge [\![\varphi]\!](t) - [\![\varphi]\!](s) \\ \forall \varphi \in \text{CLQMU}\setminus\{\exists\ \ \} & bd^{\mathrm{As}}(s,t) \ge [\![\varphi]\!](t) - [\![\varphi]\!](s) \\ \forall \varphi \in \text{CLQMU}\setminus\{D(r,c)\} & bd^{\mathrm{Sa}}(s,t) \ge [\![\varphi]\!](t) - [\![\varphi]\!](s) \\ \forall \varphi \in \text{CLQMU} & bd^{\mathrm{Ss}}(s,t) \ge |[\![\varphi]\!](t) - [\![\varphi]\!](s)|.\end{array}$$

As noted before, each bound of the form $d(s,t) \ge [\![\varphi]\!](t) - [\![\varphi]\!](s)$ trivially leads to a bound of the form $\overline{d}(s,t) \ge |[\![\varphi]\!](t) - [\![\varphi]\!](s)|$. The bounds are tight for finitely branching systems, and the following theorem identifies which fragments of quantitative $\mu$-calculus suffice for characterizing each branching distance. The formula scheme used to characterize $bd^{\mathrm{Ss}}$ is reminiscent of the one used in [1] for bisimulation.

*Theorem 14:* *For all finitely branching MTSs* $(S,\tau,\Sigma,[\cdot])$*, states* $s,t \in S$*, we have*

$$\begin{array}{ll}bd^{\mathrm{Aa}}(s,t) = \sup_{\varphi \in \text{CLQMU}\setminus\{\exists\ \ , D(r,c),\mu,\nu\}} & [\![\varphi]\!](t) - [\![\varphi]\!](s) \\ bd^{\mathrm{As}}(s,t) = \sup_{\varphi \in \text{CLQMU}\setminus\{\exists\ \ ,\mu,\nu\}} & [\![\varphi]\!](t) - [\![\varphi]\!](s) \\ bd^{\mathrm{Sa}}(s,t) = \sup_{\varphi \in \text{CLQMU}\setminus\{D(r,c),\mu,\nu\}} & [\![\varphi]\!](t) - [\![\varphi]\!](s) \\ bd^{\mathrm{Ss}}(s,t) = \sup_{\varphi \in \text{CLQMU}\setminus\{\mu,\nu\}} & [\![\varphi]\!](t) - [\![\varphi]\!](s).\end{array}$$

*Proof:*

*Part 1:* Consider the statement about $bd^{\mathrm{Aa}}$. For all $s \in S$, we define the sequence of formulas $(\varphi_s^k)_{k\ge 0}$ as follows.

$$\varphi_s^0 = \bigvee_{r\in\Sigma} D([s](r), r),$$
$$\varphi_s^{k+1} = \varphi_s^0 \vee \bigvee_{s'\in\tau(s)} \forall\ \ \varphi_{s'}^k.$$

First, one can easily prove by induction that, for all $k \in \mathbb{N}$ and $s \in S$, $[\![\varphi_s^k]\!](s) = 0$. Recall from Definition 13 that the distance $bd^{\mathrm{Aa}}$ is defined as the least fixpoint of $H^{\mathrm{Aa}}$. Denoting by $(H^{\mathrm{Aa}})^k$ a sequence of $k$ applications of $H^{\mathrm{Aa}}$, since the MTS is finitely branching, we have that $bd^{\mathrm{Aa}} = \lim_k (H^{\mathrm{Aa}})^k(pd)$. We prove by induction on $k$ that, for all $s, t \in S$, $[\![\varphi_s^k]\!](t) = (H^{\mathrm{Aa}})^k(pd)(s,t)$.

$$[\![\varphi_s^0]\!](t) = \max_{r\in\Sigma} d([s](r),[t](r)) \\ = pd(s,t) = (H^{\mathrm{Aa}})^0(pd)(s,t);$$

$$[\![\varphi_s^{k+1}]\!](t) = [\![\varphi_s^0]\!](t) \sqcup \max_{s'\in\tau(s)} \min_{t'\in\tau(t)} [\![\varphi_{s'}^k]\!](t') \\ = pd(s,t) \sqcup \max_{s'\in\tau(s)} \min_{t'\in\tau(t)} (H^{\mathrm{Aa}})^k(pd)(s',t') \\ = (H^{\mathrm{Aa}})^{k+1}(pd)(s,t).$$

Let $CQ = \textsc{ClQmu} \setminus \{\exists\ \ , D(r,c), \mu, \nu\}$, it follows that

$$\sup_{\varphi\in CQ} [\![\varphi]\!](t) - [\![\varphi]\!](s) \geq \sup_{k\in\mathbb{N}}\ [\![\varphi_s^k]\!](t) - [\![\varphi_s^k]\!](s) \\ = \sup_{k\in\mathbb{N}} (H^{\mathrm{Aa}})^k(pd)(s,t) - 0 \\ = bd^{\mathrm{Aa}}(s,t).$$

*Part 2:* To prove the statement concerning $bd^{\mathrm{As}}(s,t)$, we define the following sequence of formulas $(\varphi_s^k)_{k\in\mathbb{N}}$.

$$\varphi_s^0 = \bigvee_{r\in\Sigma} D([s](r),r) \vee D(r,[s](r)) \\ \varphi_s^{k+1} = \varphi_s^0 \vee \bigvee_{s'\in\tau(s)} \forall\ \ \varphi_{s'}^k.$$

We then proceed similarly to the previous part.

*Part 3:* To prove the bound on $bd^{\mathrm{Sa}}(s,t)$, we use the formulas:

$$\varphi_s^0 = \bigvee_{r\in\Sigma} D([s](r),r) \\ \varphi_s^{k+1} = \varphi_s^0 \vee \bigvee_{s'\in\tau(s)} \forall\ \ \varphi_{s'}^k \vee \exists\ \ \Big(\bigwedge_{s'\in\tau(s)} \varphi_{s'}^k\Big).$$

Once again, one can easily prove by induction that, for all $k \in \mathbb{N}$ and $s \in S$, $[\![\varphi_s^k]\!](s) = 0$. The distance $bd^{\mathrm{Sa}}$ is defined as the least fixpoint of $H^{\mathrm{Sa}}$. In particular, denoting by $(H^{\mathrm{Sa}})^k$ a sequence of $k$ applications of $H^{\mathrm{Sa}}$, again due to the fact that the MTS is finitely branching we have $bd^{\mathrm{Sa}} = \lim_k (H^{\mathrm{Sa}})^k(pd)$. We prove by induction on $k$ that, for all $s,t \in S$, $[\![\varphi_s^k]\!](t) = (H^{\mathrm{Sa}})^k(pd)(s,t)$.

$$[\![\varphi_s^0]\!](t) = \max_{r\in\Sigma} \big(d([s](r),[t](r)) \sqcup d([t](r),[s](r))\big) \\ = pd(s,t) = (H^{\mathrm{Sa}})^0(pd)(s,t);$$

$$[\![\varphi_s^{k+1}]\!](t) = [\![\varphi_s^0]\!](t) \sqcup \max_{s'\in\tau(s)} \min_{t'\in\tau(t)} [\![\varphi_{s'}^k]\!](t') \\ \sqcup \max_{t'\in\tau(t)} \min_{s'\in\tau(s)} [\![\varphi_{s'}^k]\!](t') \\ = pd(s,t) \sqcup \max_{s'\in\tau(s)} \min_{t'\in\tau(t)} (H^{\mathrm{Sa}})^k(pd)(s',t') \\ \sqcup \max_{t'\in\tau(t)} \min_{s'\in\tau(s)} (H^{\mathrm{Sa}})^k(pd)(s',t') \\ = (H^{\mathrm{Sa}})^{k+1}(pd)(s,t).$$

Let $CQ = \textsc{ClQmu} \setminus \{D(r,c), \mu, \nu\}$, it follows that

$$\sup_{\varphi\in CQ} [\![\varphi]\!](t) - [\![\varphi]\!](s) \geq \sup_{k\in\mathbb{N}}\ [\![\varphi_s^k]\!](t) - [\![\varphi_s^k]\!](s) \\ = \sup_{k\in\mathbb{N}} (H^{\mathrm{Sa}})^k(pd)(s,t) - 0 \\ = bd^{\mathrm{Sa}}(s,t).$$

*Part 4:* To prove the bound on $bd^{\mathrm{Ss}}(s,t)$, we use the formulas:

$$\varphi_s^0 = \bigvee_{r\in\Sigma} D([s](r),r) \vee D(r,[s](r)) \\ \varphi_s^{k+1} = \varphi_s^0 \vee \bigvee_{s'\in\tau(s)} \forall\ \ \varphi_{s'}^k \vee \exists\ \ \Big(\bigwedge_{s'\in\tau(s)} \varphi_{s'}^k\Big).$$

We then proceed similarly to the previous parts. ■

Again, the logical characterization above is in terms of formulas defined over a potentially uncountable set of constants: in general, we need one constant for each element of a metric space corresponding to a proposition. As in the linear case, we show that if the MTS is separable, then it suffices to consider formulas defined over the countable set of constants corresponding to the countable bases of the metric spaces for the various propositions. Similarly to the linear case, the result follows from the observation that the value of a formula, at every state, can be approximated arbitrarily well by the value of a formula containing only constants that belong to the countable bases of the metric spaces.

*Theorem 15:* *If an MTS $M = (S,\tau,\Sigma,[\cdot])$ is both finitely branching and separable, then the characterizations provided by Theorem 14 hold also when we restrict the formulas of quantitative $\mu$-calculus to those that contain only constants from the countable set $\bigcup_{r\in\Sigma} \mathcal{B}_r$, where $\mathcal{B}_r$ is a countable basis for the metric space $(X_r, d_r)$, for each $r \in \Sigma$.*

### *D. Computing the Branching Distances*

Given a finite MTS $M = (S,\tau,\Sigma,[\cdot])$ and $x \in \{\mathrm{Ss},\mathrm{Sa},\mathrm{As},\mathrm{Aa}\}$, we can compute $bd^x(s,t)$ for all states $s,t \in S$ by computing in an iterative fashion the fixpoints of Definition 13. Precisely, we let, for all $s,t \in S$ and all $k \geq 0$:

$$d^0(s,t) = 0 \\ d^{k+1}(s,t) = pd(s,t) \sqcup \max_{s'\in\tau(s)} \min_{t'\in\tau(t)} d^k(s',t'). \qquad (1)$$

Then $bd^{\mathrm{Aa}} = \lim_{k\to\infty} d^k$. The following theorem shows that the above iteration converges in at most $|S|^2$ steps.

*Theorem 16:* *For all MTSs $M$ having $n$ states and $m$ edges, the iteration (1) converges in at most $n^2$ steps.*

*Proof:* The computation of (1) is equivalent to solve a maximum-value-reachability game having state space $S \times S$ and, for each state $(s,t) \in S \times S$, set of moves $\tau(s)$ for Player 1, and $\tau(t)$ for Player 2. The pair of moves $(s',t')$ from $(s,t)$ leads to state $(s',t')$ of the game. Every state $(s,t)$ of the game has value $pd(s,t)$, and the goal for Player 1 is to maximize the value reached along a play of the game. It is then easy to prove by induction that $d^k(s,t)$ represents the

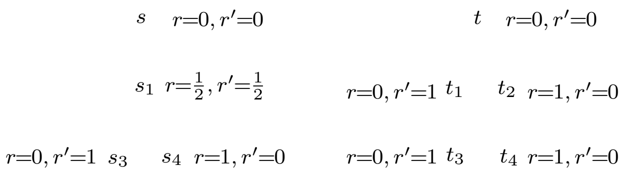
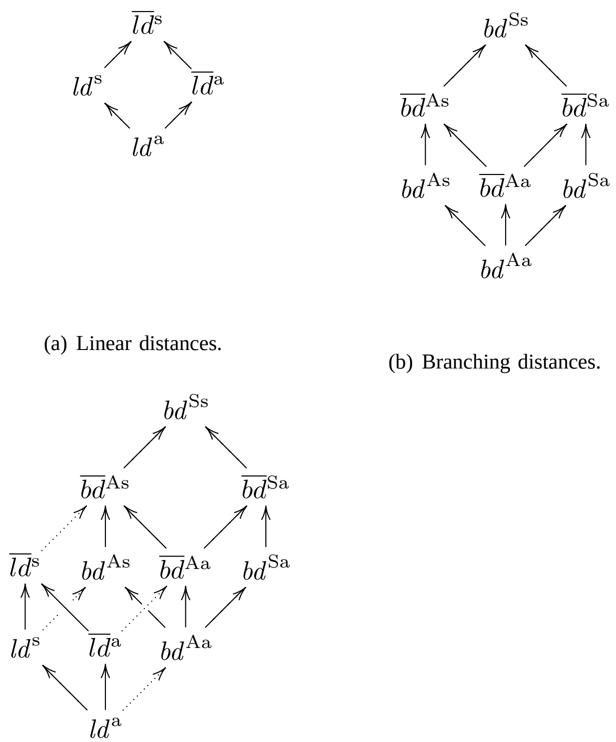



Fig. 5. Linear versus branching distances on a deterministic MTS.



(a) Linear distances.

(b) Branching distances.

(c) All distances.

Fig. 6. Relations between distances, where $f \to g$ means $f \leq g$. In (c), the dotted arrows collapse to equality for boolean, deterministic MTSs.

maximum value Player 1 can ensure in at most $k$ steps. Let $Z = \{pd(s,t) \mid (s,t) \in S \times S\}$, and for $z \in Z$ let $T_{\geq z} = \{(s,t) \in S \times S \mid pd(s,t) \geq z\}$. For $z \in Z$, assume that from a state $(s,t)$ Player 1 can force the game to $T_{\geq z}$. Then, the value of the game from $(s,t)$ for Player 1 is at least $z$; moreover, $T_{\geq z}$ can be reached in at most $n^2$ steps, as this is a standard graph reachability game. If on the other hand Player 1 cannot force the game to $T_{\geq z}$ from $(s,t)$, by determinacy of reachability games Player 2 has a strategy to keep the game always in $T_{<z} = S \times S \setminus T_{\geq z}$, and the value of the game from $(s,t)$ will be below $z$. Let $z(s,t)$ be the highest $z \in Z$ for which Player 1 can force the game to $T_{\geq z}$. From the above analysis we have that $z(s,t)$ is the value of the game at $(s,t)$; moreover, this value is attainable in at most $n^2$ steps. Together with the characterization of $d^k$, this shows that the sequence $\big(d^k(s,t)\big)_{k\geq 0}$ converges in at most $n^2$ steps. ■

In an MTS with $n$ states and $m$ edges, each step of (1) can be done in $O(n \cdot m)$ time, since there are $O(n \cdot m)$ edges in the product game. This yields a complexity of $O(n^3 \cdot m)$.

## V. Comparing the Linear and Branching Distances

In this section, we provide a comparison between linear and branching distances. Just as similarity implies trace inclusion, we have both $ld^{\mathrm{a}} \leq bd^{\mathrm{Aa}}$ and $ld^{\mathrm{s}} \leq \overline{bd}^{\mathrm{As}}$; just as bisimilarity implies trace equivalence, we have $\overline{ld}^{\mathrm{s}} \leq bd^{\mathrm{Ss}}$ and $\overline{ld}^{\mathrm{a}} \leq \overline{bd}^{\mathrm{Sa}}$. Moreover, in the non-quantitative setting, trace inclusion (resp. trace equivalence) coincides with (bi-)similarity on deterministic systems. This result generalizes to distances over MTSs that are both deterministic and boolean, but not to distances over MTSs that are just deterministic. To formalize these results, we say that an MTS is *boolean* if all its propositions are evaluated in the metric space $\mathbf{X}_{\mathbb{B}}$.

*Theorem 17: The following properties hold.*

1) *For all MTSs, we have*

$$ld^{\mathrm{a}} \leq bd^{\mathrm{Aa}} \quad ld^{\mathrm{s}} \leq \overline{bd}^{\mathrm{As}} \quad \overline{ld}^{\mathrm{a}} \leq \overline{bd}^{\mathrm{Sa}} \quad \overline{ld}^{\mathrm{s}} \leq bd^{\mathrm{Ss}}.$$

*Moreover, the inequalities cannot be replaced by equalities.*

2) *For all boolean, deterministic MTSs we have*

$$ld^{\mathrm{a}} = bd^{\mathrm{Aa}} \quad ld^{\mathrm{s}} = bd^{\mathrm{As}} \quad \overline{ld}^{\mathrm{a}} = \overline{bd}^{\mathrm{Aa}} \quad \overline{ld}^{\mathrm{s}} = \overline{bd}^{\mathrm{As}}.$$

*These equalities need not to hold for non-boolean, deterministic MTSs.*

The relations of Part 1 are illustrated in Figure 6(c).

*Proof: Statement 1.* We prove $ld^{\mathrm{a}} \leq bd^{\mathrm{Aa}}$, the other cases being similar. First, we note that $bd^{\mathrm{Aa}}(s,t) \leq c$ iff

$$\forall \epsilon' > 0 \,.\, \forall s' \in \tau(s) \,.\, \exists t' \in \tau(t) \,.\, bd^{\mathrm{Aa}}(s',t') \leq c + \epsilon'. \quad (*)$$

Let $s,t \in S$ be states and let $\epsilon > 0$. We show that $ld^{\mathrm{a}}(s,t) \leq bd^{\mathrm{Aa}}(s,t) + \epsilon$. We do so by demonstrating that $ld^{\mathrm{a}}(\sigma,t) := \inf_{\rho \in \mathit{Tr}(t)} td(\sigma,\rho) \leq bd^{\mathrm{Aa}}(s,t) + \epsilon$ for all $\sigma \in \mathit{Tr}(s)$.

Let $\sigma = s_0 s_1 s_2 \ldots$ be a trace in $s$. We build a trace $\rho^* = t_0 t_1 t_2 \ldots$ in $\mathit{Tr}(t)$ as follows. We have $t_0 = t$ and, for all $i \geq 0$, $t_{i+1}$ is such that

$$bd^{\mathrm{Aa}}(s_{i+1}, t_{i+1}) \leq bd^{\mathrm{Aa}}(s,t) + \sum_{j=1}^{i+1} \frac{\epsilon}{2^j}.$$

We show by induction that $t_i$ is well-defined. Clearly, $t_0$ is well-defined. Assume that $t_i$ is well-defined. Then $bd^{\mathrm{Aa}}(s_i,t_i) \leq bd^{\mathrm{Aa}}(s,t) + \sum_{j=1}^{i} \frac{\epsilon}{2^j}$. We obtain from (*) by taking $s = s_i$, $t = t_i$, $s' = s_{i+1}$

$$c = bd^{\mathrm{Aa}}(s,t) + \sum_{j=1}^{i} \frac{\epsilon}{2^j}$$

$$\epsilon' = \frac{\epsilon}{2^{i+1}}$$

that there exists a $t' \in \tau(t_i)$ with $bd^{\mathrm{Aa}}(s_{i+1},t') \leq bd^{\mathrm{Aa}}(s,t) + \sum_{j=1}^{i} \frac{\epsilon}{2^j} + \frac{\epsilon}{2^{i+1}} = bd^{\mathrm{Aa}}(s,t) + \sum_{j=1}^{i+1} \frac{\epsilon}{2^j}$. We take $t_{i+1} = t'$. Then,

$$\begin{aligned} ld^{\mathrm{a}}(\sigma,t) &= \inf_{\rho \in \mathit{Tr}(t)} td(\sigma,\rho) \\ &\le td(\sigma,\rho^*) \\ &= \sup_{i\in\mathbb{N}} pd(\sigma_i,\rho_i) \\ &\le \sup_{i\in\mathbb{N}} bd^{\mathrm{Aa}}(\sigma_i,\rho_i) \\ &\le bd^{\mathrm{Aa}}(s,t) + \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} \qquad = bd^{\mathrm{Aa}}(s,t) + \epsilon. \end{aligned}$$

*Statement 2.* Let $M = (S, \tau, \Sigma, [\cdot])$ be a boolean, deterministic MTS, and let $s, t \in S$ be states. We show that $ld^{\mathrm{a}} = bd^{\mathrm{Aa}}$. The other cases are similar. By Part 1 of this theorem, we know that $ld^{\mathrm{a}} \le bd^{\mathrm{Aa}}$. To prove that $ld^{\mathrm{a}} \ge bd^{\mathrm{Aa}}$, we show that $H^{\mathrm{Aa}}(ld^{\mathrm{a}}) = ld^{\mathrm{a}}$, i.e. that $ld^{\mathrm{a}}$ is a fixpoint of $H^{\mathrm{Aa}}$. As $bd^{\mathrm{Aa}}$ is the least fixpoint of $H^{\mathrm{Aa}}$, we obtain $ld^{\mathrm{a}} \ge bd^{\mathrm{Aa}}$. First, we observe that

$$\begin{aligned} & H^{\mathrm{Aa}}(ld^{\mathrm{a}})(s,t) \\ &= pd(s,t) \sqcup \sup_{s'\in\tau(s)} \inf_{t'\in\tau(t)} ld^{\mathrm{a}}(s',t') \\ &= pd(s,t) \sqcup \sup_{s'\in\tau(s)} \inf_{t'\in\tau(t)} \sup_{\sigma'\in \mathit{Paths}(s')} \inf_{\rho'\in \mathit{Paths}(t')} td(\sigma',\rho') \\ &\ge pd(s,t) \sqcup \sup_{s'\in\tau(s)} \sup_{\sigma'\in \mathit{Paths}(s')} \inf_{t'\in\tau(t)} \inf_{\rho'\in \mathit{Paths}(t')} td(\sigma',\rho') \\ &= \sup_{\sigma\in \mathit{Paths}(s)} \inf_{\rho\in \mathit{Paths}(t)} td(\sigma,\rho) \\ &= ld^{\mathrm{a}}(s,t). \end{aligned}$$

So $H^{\mathrm{Aa}}(ld^{\mathrm{a}})(s,t) \ge ld^{\mathrm{a}}(s,t)$. We show that also $H^{\mathrm{Aa}}(ld^{\mathrm{a}})(s,t) \le ld^{\mathrm{a}}(s,t)$. If $pd(s,t) = 1$, then $H^{\mathrm{Aa}}(ld^{\mathrm{a}})(s,t) = ld^{\mathrm{a}}(s,t) = 1$. Hence, assume $pd(s,t) = 0$. We distinguish two cases.

*Case 1:* $\sup_{s'\in\tau(s)} \inf_{t'\in\tau(t)} pd(s',t') = 1$. Then one easily shows that $H^{\mathrm{Aa}}(ld^{\mathrm{a}})(s,t) = 1 = ld^{\mathrm{a}}(s,t)$.

*Case 2:* $\sup_{s'\in\tau(s)} \inf_{t'\in\tau(t)} pd(s',t') = 0$. Since $M$ is deterministic and boolean, we know that for all $s' \in \tau(s)$, there is a $t_{s'} \in \tau(t)$ such that $pd(s', t_{s'}) = 0$ and $pd(s',t') = 1$ for $t' \ne t_{s'}$. Then, we have for all $s' \in \tau(s), t' \in \tau(t), t' \ne t_{s'}$, $\sigma' \in \mathit{Paths}(s')$, $\rho' \in \mathit{Paths}(t')$, and $\rho_{s'} \in \mathit{Paths}(t_{s'})$ that

$$td(\sigma', \rho_{t_{s'}}) \le 1 \quad \text{and} \quad td(\sigma',\rho') = 1$$

and therefore

$$\inf_{\rho'\in \mathit{Paths}(t_{s'})} td(\sigma',\rho') \le \inf_{\rho'\in \mathit{Paths}(t')} td(\sigma',\rho')$$

so

$$\inf_{\rho'\in \mathit{Paths}(t_{s'})} td(\sigma',\rho') \le \inf_{t'\in\tau(t)} \inf_{\rho'\in \mathit{Paths}(t')} td(\sigma',\rho'). \qquad (2)$$

Recalling that $pd(s,t) = 0$, we get

$$\begin{aligned} & H^{\mathrm{Aa}}(ld^{\mathrm{a}})(s,t) \\ &= \sup_{s'\in\tau(s)} \inf_{t'\in\tau(t)} \sup_{\sigma'\in \mathit{Paths}(s')} \inf_{\rho'\in \mathit{Paths}(t')} td(\sigma',\rho') \\ &\le \sup_{s'\in\tau(s)} \sup_{\sigma'\in \mathit{Paths}(s')} \inf_{\rho'\in \mathit{Paths}(t_{s'})} td(\sigma',\rho') \qquad \text{by (2)} \\ &\le \sup_{s'\in\tau(s)} \sup_{\sigma'\in \mathit{Paths}(s')} \inf_{t'\in\tau(t)} \inf_{\rho'\in \mathit{Paths}(t')} td(\sigma',\rho') \\ &= \sup_{\sigma\in \mathit{Paths}(s)} \inf_{\rho\in \mathit{Paths}(t)} td(\sigma,\rho) = ld^{\mathrm{a}}(s,t). \end{aligned}$$

To see that the equalities need not hold for non-boolean, deterministic MTSs, consider the MTS in Figure 5. We have $ld^x(s,t) = \frac{1}{2}$, while $bd^x(s,t) = 1$. ■

## VI. DISCOUNTING

Our theory can also be developed in a *discounted* version, in which distances occurring $i$ steps in the future are multiplied by $\alpha^i$, where $\alpha$ is a discount factor in $(0,1]$. This discounted setting is common in the theory of games (see e.g. [9]) and optimal control (see e.g. [7]), and it leads to robust theories of quantitative systems [4]. In the discouned setting, behavioral differences arising far into the future are given less relative weight than behavioral differences affecting the present or the near future. Hence, the discounted setting leads to notions of "local similarity" that enjoy many pleasant mathematical properties.

### A. Discounted Linear Distances and Logics

The basic ingredient of the discounted version of the linear theory is the following discounted trace distance.

*Definition 14:* **(discounted trace distance)** Let $\alpha \in (0,1]$. We define the *$\alpha$-discounted trace distance* $td_\alpha : \mathcal{U}[\Sigma]^\omega \times \mathcal{U}[\Sigma]^\omega \to \mathbb{R}$ by letting, for $\sigma, \rho \in \mathcal{U}[\Sigma]^\omega$, $td_\alpha(\sigma,\rho) = \sup_{i\in\mathbb{N}} \alpha^i pd(\sigma_i,\rho_i)$. ■

For all discount factors $\alpha \in (0,1]$, the discounted linear distances $ld^{\mathrm{a}}_\alpha$ and $ld^{\mathrm{s}}_\alpha$ can be defined as in Definition 12, by simply replacing $td$ with $td_\alpha$.

In order to define an LTL-like logic that characterizes the above distances, given $\alpha \in (0,1]$, we parametrize each temporal operator from QLTL with a (possibly different) discount factor $\beta \le \alpha$, thus obtaining the logic $\mathrm{QLTL}_\alpha$. Formally, formulas from $\mathrm{QLTL}_\alpha$ are generated by the following grammar:

$$\varphi ::= D(r,c) \mid D(c,r) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \quad {}_\beta\varphi \mid \Diamond_\beta \varphi \mid \Box_\beta \varphi$$

where $r \in \Sigma$ is a proposition, $c \in \bigcup_{r\in\Sigma} X_r$ is a constant, and $\beta \in (0,\alpha]$ is a discount factor. The semantics of $\mathrm{QLTL}_\alpha$ is the same as the one of QLTL, except for the discounted operators:

$$\begin{aligned} [\![ \quad {}_\beta\varphi]\!](\sigma) &= \beta\, [\![\varphi]\!](\sigma^1) \\ [\![\Diamond_\beta\varphi]\!](\sigma) &= \sup\{\beta^i\, [\![\varphi]\!](\sigma^i) \mid i \ge 0\} \\ [\![\Box_\beta\varphi]\!](\sigma) &= \inf\{\beta^i\, [\![\varphi]\!](\sigma^i) \mid i \ge 0\}. \end{aligned}$$

All theorems that were proven for the linear distances and QLTL have a corresponding discounted version, that applies to the discounted distances and $\mathrm{QLTL}_\alpha$. For instance, computing the discounted linear distance between all pairs of states in a finite MTS is still PSPACE-complete. Also, we have the following characterization, analogue to Theorem 5.

*Theorem 18:* *If an MTS $M = (S, \tau, \Sigma, [\cdot])$ is finitely branching, then we have that for all $\alpha \in (0,1]$, $s,t \in S$:*

$$\begin{aligned}
ld^{\mathrm{a}}_\alpha(s,t) &= \sup_{\varphi \in \mathrm{Q}\textsc{ltl}_\alpha \setminus \{D(r,c), \Diamond, \Box\}} [\![\varphi]\!](t) - [\![\varphi]\!](s) \\
\overline{ld}^{\mathrm{a}}_\alpha(s,t) &= \sup_{\varphi \in \mathrm{Q}\textsc{ltl}_\alpha \setminus \{D(r,c), \Diamond, \Box\}} |[\![\varphi]\!](t) - [\![\varphi]\!](s)| \\
ld^{\mathrm{s}}_\alpha(s,t) &= \sup_{\varphi \in \mathrm{Q}\textsc{ltl}_\alpha \setminus \{\Diamond, \Box\}} [\![\varphi]\!](t) - [\![\varphi]\!](s) \\
\overline{ld}^{\mathrm{s}}_\alpha(s,t) &= \sup_{\varphi \in \mathrm{Q}\textsc{ltl}_\alpha \setminus \{\Diamond, \Box\}} |[\![\varphi]\!](t) - [\![\varphi]\!](s)|.
\end{aligned}$$

### B. Discounted Branching Distances and Logics

Similarly to the linear case, we can define the following discounted branching distances.

*Definition 15:* **(discounted branching distances)** For $\alpha \in (0,1]$ and $x \in \{\mathrm{Aa}, \mathrm{As}, \mathrm{Sa}, \mathrm{Ss}\}$, consider the four operators $H^x_\alpha : (S^2 \to \mathrm{I\!R}) \to (S^2 \to \mathrm{I\!R})$ defined as follows, for $d : S^2 \to \mathrm{I\!R}$:

$$\begin{aligned}
H^{\mathrm{Aa}}_\alpha(d)(s,t) &= pd(s,t) \sqcup \alpha \sup_{s' \in \tau(s)} \inf_{t' \in \tau(t)} d(s',t') \\
H^{\mathrm{As}}_\alpha(d)(s,t) &= \overline{pd}(s,t) \sqcup \alpha \sup_{s' \in \tau(s)} \inf_{t' \in \tau(t)} d(s',t') \\
H^{\mathrm{Sa}}_\alpha(d)(s,t) &= pd(s,t) \sqcup \alpha \sup_{s' \in \tau(s)} \inf_{t' \in \tau(t)} d(s',t') \\
&\qquad \sqcup \alpha \sup_{t' \in \tau(t)} \inf_{s' \in \tau(s)} d(s',t') \\
H^{\mathrm{Ss}}_\alpha(d)(s,t) &= \overline{pd}(s,t) \sqcup \alpha \sup_{s' \in \tau(s)} \inf_{t' \in \tau(t)} d(s',t') \\
&\qquad \sqcup \alpha \sup_{t' \in \tau(t)} \inf_{s' \in \tau(s)} d(s',t').
\end{aligned}$$

For $x \in \{\mathrm{Aa}, \mathrm{As}, \mathrm{Sa}, \mathrm{Ss}\}$, we define the *$\alpha$-discounted branching distance $bd^x_\alpha$* as the least fixpoint of the operator $H^x_\alpha$. ■

Given a finite MTS, the discounted branching distance between all pairs of states can be computed in polynomial time as explained in Section IV-D.

Next, we introduce *discounted quantitative $\mu$-calculus*, whose syntax is the same as the one of quantitative $\mu$-calculus, except that the "next" operator is parametrized by a discount factor. Formally, for all $\alpha \in (0,1]$, formulas in $\mathrm{Q}\textsc{mu}_\alpha$ are generated by the grammar:

$$\begin{aligned}
\varphi ::= {} & D(r,c) \mid D(c,r) \mid x \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists \ \ _\beta\varphi \mid \forall \ \ _\beta\varphi \\
& \mid \mu x \,.\, \varphi \mid \nu x \,.\, \varphi
\end{aligned}$$

for propositions $r \in \Sigma$, variables $x \in V$, constants $c \in \bigcup_{r \in \Sigma} X_r$, and discount factors $\beta \in (0, \alpha]$. The semantics of $\mathrm{Q}\textsc{mu}_\alpha$ coincides with the one of $\mathrm{Q}\textsc{mu}$ (see Section IV-B) except for:

$$\begin{aligned}
[\![\exists \ \ _\beta\varphi]\!]_{\mathcal{E}}(s) &= \beta \sup_{s' \in \tau(s)} [\![\varphi]\!]_{\mathcal{E}}(s') \\
[\![\forall \ \ _\beta\varphi]\!]_{\mathcal{E}}(s) &= \beta \inf_{s' \in \tau(s)} [\![\varphi]\!]_{\mathcal{E}}(s').
\end{aligned}$$

We denote $\mathrm{C}\textsc{l}\mathrm{Q}\textsc{mu}_\alpha$ the fragment of $\mathrm{Q}\textsc{mu}_\alpha$ containing only closed formulas. Again, we have the following characterization, analogue to Theorem 14.

*Theorem 19:* *For all finitely branching MTSs $(S, \tau, \Sigma, [\cdot])$, states $s, t \in S$, and discount factors $\alpha \in (0,1]$, we have*

$$\begin{aligned}
bd^{\mathrm{Aa}}_\alpha(s,t) &= \sup\nolimits_{\varphi \in \mathrm{C}\textsc{l}\mathrm{Q}\textsc{mu}_\alpha \setminus \{\exists \ \ , D(r,c), \mu, \nu\}} && [\![\varphi]\!](t) - [\![\varphi]\!](s) \\
bd^{\mathrm{As}}_\alpha(s,t) &= \sup\nolimits_{\varphi \in \mathrm{C}\textsc{l}\mathrm{Q}\textsc{mu}_\alpha \setminus \{\exists \ \ , \mu, \nu\}} && [\![\varphi]\!](t) - [\![\varphi]\!](s) \\
bd^{\mathrm{Sa}}_\alpha(s,t) &= \sup\nolimits_{\varphi \in \mathrm{C}\textsc{l}\mathrm{Q}\textsc{mu}_\alpha \setminus \{D(r,c), \mu, \nu\}} && [\![\varphi]\!](t) - [\![\varphi]\!](s) \\
bd^{\mathrm{Ss}}_\alpha(s,t) &= \sup\nolimits_{\varphi \in \mathrm{C}\textsc{l}\mathrm{Q}\textsc{mu}_\alpha \setminus \{\mu, \nu\}} && [\![\varphi]\!](t) - [\![\varphi]\!](s).
\end{aligned}$$

## VII. Conclusions

In this paper, we have provided metric extensions of the classical linear and branching relations: trace inclusion, trace equivalence, simulation, and bisimulation. We remark that, while metric analogues of bisimulation had been known for some time [8], [17], this is not the case for the other notions, which had escaped attention thus far; [6] extends the results in the present paper to the setting of concurrent, stochastic games.

We hope that the introduction of these quantitative asymmetrical and symmetrical distances constitutes a useful step toward a *quantitative theory of systems*, in which the classical boolean setting of specification and verification is replaced by a setting in which properties have (real-valued, or metric) values, and verification can yield not only yes/no answers, but also measures of quality, adequacy, and cost.

We have provided three main classes of characterizations for linear and branching distances:

1) *Distances as upper bounds for logic valuations.* Results in this class state that the distances provide an upper bound for the difference in value of formulas of linear (QLTL) and branching (QMU) logics. Results of this type are Theorems 4 and 13.
2) *Logics as full characterizations of distances.* Results in this class state that the distances are equal to the supremum of the difference in value of all linear, or branching formulas. Results of this type are Theorems 5 and 14.
3) *Relations among distances.* Results in this class compare the value of linear and branching distances; results of this type are Theorems 2, 11, and 17.

Results in classes 1 and 3 hold for general MTSs, and are thus particularly satisfying. In contrast, as we have seen, results in class 2 hold only for finitely branching MTSs. Many MTSs of interest are not finitely branching: for instance, in a hybrid system, there can be uncountably many successors of a state, corresponding to the real-valued length of time steps possible from the state. It is an interesting open problem to investigate classes of MTSs that are more general than finitely branching MTSs, and for which results of class 2 still hold.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